Appendix D: Basic Matrix Algebra

This appendix briefly reviews basic matrix algebra from a perspective of this book. The presentation presupposes you are familiar with the concepts. You may need to review your mathematics book for additional details.

D.1 Basic Definitions

A rectangular array of numbers is called a matrix. The matrix shown in Equation (D.1) has m rows and n columns. The size of the matrix is said to be (m x n).The element in the ith row and j^{th} column is represented by a_{ij}.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{m1} & a_{m2} & \bullet & \bullet & a_{mn} \end{bmatrix} \qquad \textbf{(D.1)}$$

D.2 Addition of Matrices

Addition of matrices can only be performed for matrices having the same number of rows and columns. The sum of two matrices [A] and [B] of m rows and n columns results in a matrix [C] of m rows and n columns and is represented by Equation (D.2a).

$$[C] = [A] + [B] \qquad \textbf{(D.2a)}$$

The elements of the matrix [C] can be found using Equation (D.2b).

$$c_{ij} = a_{ij} + b_{ij} \qquad \begin{matrix} i = 1, 2 \cdot \quad \cdot \quad \cdot m \\ j = 1, 2 \cdot \quad \cdot \quad \cdot n \end{matrix} \qquad \textbf{(D.2b)}$$

D.3 Multiplication of Matrices

Multiplication of a matrix by a number results in a matrix where all elements are multiplied by the number as shown in Equation (D.3).

$$q[A] = q\begin{bmatrix} a_{11} & a_{12} & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{m1} & a_{m2} & \bullet & \bullet & a_{mn} \end{bmatrix} = \begin{bmatrix} qa_{11} & qa_{12} & \bullet & \bullet & qa_{1n} \\ qa_{21} & qa_{22} & \bullet & \bullet & qa_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ qa_{m1} & a_{m2} & \bullet & \bullet & qa_{mn} \end{bmatrix} \qquad \textbf{(D.3)}$$

The order of multiplication is important when two matrices are multiplied. In Equation (D.4a), matrix [A] is said to pre-multiply matrix [B] and matrix [B] is said to post-multiply matrix [A].

$$[C] = [A][B] \qquad \textbf{(D.4a)}$$

In Equation (D.4a) the number of columns of matrix [A] must equal to number of rows of matrix [B]. If matrix [A] of size (m x n) pre-multiplies matrix [B] of size (n x p), the result is a matrix [C] of size (m x p). The elements of matrix [C] can be found from

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad \begin{matrix} i = 1, 2 \cdot \quad \cdot \quad \cdot m \\ j = 1, 2 \cdot \quad \cdot \quad \cdot p \end{matrix} \qquad \textbf{(D.4b)}$$

D.4 Matrix and its Transpose

The transpose of a rectangular matrix [A] consisting of m rows and n columns is written as $[A]^T$ and are related as shown in Equation (D.5).

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{m1} & a_{m2} & \bullet & \bullet & a_{mn} \end{bmatrix} \qquad [A]^T = \begin{bmatrix} a_{11} & a_{21} & \bullet & \bullet & a_{m1} \\ a_{12} & a_{22} & \bullet & \bullet & a_{m2} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{1n} & a_{2n} & \bullet & \bullet & a_{mn} \end{bmatrix} \qquad \textbf{(D.5)}$$

The element a_{ij} of matrix [A] becomes element a_{ji} in the transposed matrix $[A]^T$.

A *square* matrix (same number of rows and columns) is said to be symmetric if the transpose of the matrix is the same as the original matrix as shown in Equation (D.6)

$$\text{Symmetric Matrix} \qquad [A]^T = [A] \qquad \textbf{(D.6)}$$

Equation (D.7) lists the rules that apply to transpose of matrices during addition and multiplications.

$$([A] + [B])^T = [A]^T + [B]^T \qquad ([A][B])^T = [B]^T[A]^T \qquad \textbf{(D.7)}$$

D.5 Determinant of a Matrix

Determinant is defined only for a square matrix and is represented as shown in Equation (D.8).

$$|A| = \det[A] = \begin{vmatrix} a_{11} & a_{12} & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{n1} & a_{n2} & \bullet & \bullet & a_{nn} \end{vmatrix} \quad \textbf{(D.8)}$$

The minor M_{ij} associated with a element a_{ij} is the determinant of the matrix in which the i^{th} row and j^{th} column have been removed. The determinant of a matrix can be found using Equation (D.9) where i is any row in the matrix or it can be found using Equation (D.10) where j is any column in the matrix.

$$|A| = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} M_{ik} \quad \textbf{(D.9)}$$

$$|A| = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} M_{kj} \quad \textbf{(D.10)}$$

If the determinant of a matrix is zero i.e., $|A| = 0$ then the matrix [A] is said to be singular. In a singular matrix either all rows are not independent or all columns are not independent.

D.6 Cramer's Rule

Cramer's rule can be used for solving a set of linear algebraic equations. Consider the set of n linear algebraic equations in matrix form shown in Equation (D.11).

$$\begin{bmatrix} a_{11} & a_{12} & \bullet & a_{1j} & \bullet & a_{1n} \\ a_{21} & a_{22} & \bullet & a_{2j} & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & & \bullet & \bullet \\ \bullet & \bullet & \bullet & a_{jj} & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{n1} & a_{n2} & \bullet & a_{nj} & \bullet & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \\ x_j \\ \\ x_n \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ \\ r_j \\ \\ r_n \end{Bmatrix} \quad \textbf{(D.11)}$$

By Cramer's rule the j^{th} unknown x_j can be found by first replacing the j^{th} column by the right hand side vector, taking the determinant of the resulting matrix, and then dividing by the determinant of the matrix [A] as shown in Equation (D.12)

$$x_j = \frac{\begin{vmatrix} a_{11} & a_{12} & \bullet & r_1 & \bullet & a_{1n} \\ a_{21} & a_{22} & \bullet & r_2 & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & & \bullet & \bullet \\ \bullet & \bullet & \bullet & r_j & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{n1} & a_{n2} & \bullet & r_n & \bullet & a_{nn} \end{vmatrix}}{|A|} \qquad j = 1, 2 \cdot \quad \cdot \quad \cdot n \tag{D.12}$$

D.7 Inverse of a Matrix

Inverse of a matrix can be found only of a square matrix. The inverse of a matrix [A] is denoted by $[A]^{-1}$. The product of a matrix and its inverse results in an identity matrix [I] as shown in Equation (D.13). The identity matrix [I] has one for the diagonal elements and all off-diagonal elements are zero.

$$[A]^{-1}[A] = [A][A]^{-1} = [I] \tag{D.13}$$

Equation (D.11) in matrix form can be written as Equation (D.14a).

$$[A]\{x\} = \{r\} \tag{D.14a}$$

where, {x} represents the unknown vector with components x_j and {r} represents the right hand side vector with components r_j. By pre-multiplying by [A]-1 to both sides of Equation (D.14a) and using Equation (D.13) we obtain the unknown vector as shown in Equation (D.14b)

$$[A]^{-1}[A]\{x\} = [A]^{-1}\{r\} \qquad \text{or} \qquad [I]\{x\} = [A]^{-1}\{r\} \qquad \text{or}$$

$$\{x\} = [A]^{-1}\{r\} \tag{D.14b}$$