Appendix D: Basic Matrix Algebra

This appendix briefly reviews basic matrix algebra from a perspective of this book. The presentation presupposes you are familiar with the concepts. You may need to review your mathematics book for additional details.

D.1 Basic Definitions

A rectangular array of numbers is called a matrix. The matrix shown in Equation (D.1) has m rows and n columns. The size of the matrix is said to be (m x n). The element in the ith row and jth column is represented by \( a_{ij} \).

\[
[A] = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\] (D.1)

D.2 Addition of Matrices

Addition of matrices can only be performed for matrices having the same number of rows and columns. The sum of two matrices \([A]\) and \([B]\) of m rows and n columns results in a matrix \([C]\) of m rows and n columns and is represented by Equation (D.2a).

\[
[C] = [A] + [B]
\] (D.2a)

The elements of the matrix \([C]\) can be found using Equation (D.2b).

\[
c_{ij} = a_{ij} + b_{ij} \quad i = 1, 2 \cdot \cdot \cdot m \\
j = 1, 2 \cdot \cdot \cdot n
\] (D.2b)

D.3 Multiplication of Matrices

Multiplication of a matrix by a number results in a matrix where all elements are multiplied by the number as shown in Equation (D.3).

\[
q[A] = q\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} = \begin{bmatrix}
qa_{11} & qa_{12} & \cdots & qa_{1n} \\
qa_{21} & qa_{22} & \cdots & qa_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
qa_{m1} & qa_{m2} & \cdots & qa_{mn}
\end{bmatrix}
\] (D.3)
The order of multiplication is important when two matrices are multiplied. In Equation (D.4a), matrix \([A]\) is said to pre-multiply matrix \([B]\) and matrix \([B]\) is said to post-multiply matrix \([A]\).

\[
[C] = [A][B] \tag{D.4a}
\]

In Equation (D.4a) the number of columns of matrix \([A]\) must equal to number of rows of matrix \([B]\). If matrix \([A]\) of size \((m \times n)\) pre-multiplies matrix \([B]\) of size \((n \times p)\), the result is a matrix \([C]\) of size \((m \times p)\). The elements of matrix \([C]\) can be found from

\[
c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad i = 1, 2, \ldots, m \quad j = 1, 2, \ldots, p \tag{D.4b}
\]

### D.4 Matrix and its Transpose

The transpose of a rectangular matrix \([A]\) consisting of \(m\) rows and \(n\) columns is written as \([A]^T\) and are related as shown in Equation (D.5).

\[
[A] = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} \quad [A]^T = \begin{bmatrix}
a_{11} & a_{21} & \cdots & a_{m1} \\
a_{12} & a_{22} & \cdots & a_{m2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & \cdots & a_{mn}
\end{bmatrix} \tag{D.5}
\]

The element \(a_{ij}\) of matrix \([A]\) becomes element \(a_{ji}\) in the transposed matrix \([A]^T\).

A *square* matrix (same number of rows and columns) is said to be symmetric if the transpose of the matrix is the same as the original matrix as shown in Equation (D.6)

\[
\text{Symmetric Matrix} \quad [A]^T = [A] \tag{D.6}
\]

Equation (D.7) lists the rules that apply to transpose of matrices during addition and multiplications.

\[
([A] + [B])^T = [A]^T + [B]^T \quad ([A][B])^T = [B]^T[A]^T \tag{D.7}
\]
D.5 Determinant of a Matrix

Determinant is defined only for a square matrix and is represented as shown in Equation (D.8).

\[
|A| = \det[A] = \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\]  \hspace{1cm} (D.8)

The minor \( M_{ij} \) associated with a element \( a_{ij} \) is the determinant of the matrix in which the \( i^{th} \) row and \( j^{th} \) column have been removed. The determinant of a matrix can be found using Equation (D.9) where \( i \) is any row in the matrix or it can be found using Equation (D.10) where \( j \) is any column in the matrix.

\[
|A| = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} M_{ik} \hspace{1cm} (D.9)
\]

\[
|A| = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} M_{kj} \hspace{1cm} (D.10)
\]

If the determinant of a matrix is zero i.e., \( |A| = 0 \) then the matrix \([A]\) is said to be singular. In a singular matrix either all rows are not independent or all columns are not independent.

D.6 Cramer’s Rule

Cramer’s rule can be used for solving a set of linear algebraic equations. Consider the set of \( n \) linear algebraic equations in matrix form shown in Equation (D.11).

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_j \\
x_{n-1} \\
x_n
\end{bmatrix}
= 
\begin{bmatrix}
r_1 \\
r_2 \\
\vdots \\
r_j \\
r_{n-1} \\
r_n
\end{bmatrix}
\]  \hspace{1cm} (D.11)

By Cramer’s rule the \( j^{th} \) unknown \( x_j \) can be found by first replacing the \( j^{th} \) column by the right
hand side vector, taking the determinant of the resulting matrix, and then dividing by the determin-

ant of the matrix \([A]\) as shown in Equation (D.12)

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & r_1 & a_{1n} \\
  a_{21} & a_{22} & \cdots & r_2 & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{n1} & a_{n2} & \cdots & r_n & a_{nn}
\end{bmatrix}
\]

\[
A_j = \frac{\begin{vmatrix}
  a_{11} & a_{12} & \cdots & r_1 & a_{1n} \\
  a_{21} & a_{22} & \cdots & r_2 & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{n1} & a_{n2} & \cdots & r_n & a_{nn}
\end{vmatrix}}{\begin{vmatrix}
  a_{11} & a_{12} & \cdots & 1 & \cdots & 0 \\
  a_{21} & a_{22} & \cdots & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & 0 & \cdots & 0
\end{vmatrix}} = \frac{\begin{vmatrix}
  a_{11} & a_{12} & \cdots & r_1 & a_{1n} \\
  a_{21} & a_{22} & \cdots & r_2 & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{n1} & a_{n2} & \cdots & r_n & a_{nn}
\end{vmatrix}}{\begin{vmatrix}
  a_{11} & a_{12} & \cdots & 1 & \cdots & 0 \\
  a_{21} & a_{22} & \cdots & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & 0 & \cdots & 0
\end{vmatrix}}
\]

\[j = 1, 2, \ldots, n\]

D.7 Inverse of a Matrix

Inverse of a matrix can be found only of a square matrix. The inverse of a matrix \([A]\) is denoted by \([A]^{-1}\). The product of a matrix and its inverse results in an identity matrix \([I]\) as shown in Equation (D.13). The identity matrix \([I]\) has one for the diagonal elements and all off-diagonal elements are zero.

\[
[A]^{-1} [A] = [A][A]^{-1} = [I]
\]

Equation (D.11) in matrix form can be written as Equation (D.14a).

\[
[A] \{x\} = \{r\}
\]

where, \(\{x\}\) represents the unknown vector with components \(x_j\) and \(\{r\}\) represents the right hand side vector with components \(r_j\). By pre-multiplying by \([A]^{-1}\) to both sides of Equation (D.14a) and using Equation (D.13) we obtain the unknown vector as shown in Equation (D.14b)

\[
[A]^{-1} [A] \{x\} = [A]^{-1} \{r\} \quad \text{or} \quad [I] \{x\} = [A]^{-1} \{r\} \quad \text{or} \quad \{x\} = [A]^{-1} \{r\}
\]