Review

Linear Strain Energy Density

\[ U_0 = \frac{1}{2} \left[ \sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \sigma_{zz} \varepsilon_{zz} + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx} \right] = \frac{1}{2} \{ \tilde{\sigma} \}^T \{ \tilde{\varepsilon} \} = \frac{1}{2} \{ \varepsilon \}^T [E] \{ \varepsilon \} \]

\[ \{ \tilde{\sigma} \} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}, \quad \{ \tilde{\varepsilon} \} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = [B] \{ d \} \]

<table>
<thead>
<tr>
<th>Axial</th>
<th>Torsion of circular shafts</th>
<th>Symmetric bending of beams</th>
</tr>
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<tbody>
<tr>
<td>[ U_a = \frac{1}{2} EA \left( \frac{du}{dx} \right)^2 ]</td>
<td>[ U_t = \frac{1}{2} GJ \left( \frac{d\phi}{dx} \right)^2 ]</td>
<td>[ U_b = \frac{1}{2} EI_{zz} \left( \frac{dv}{dx} \right)^2 ]</td>
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- Any variable that can be used for describing deformation is called the generalized displacement.
- Any variable that can be used for describing the cause that produces deformation is called the generalized force.
- Work is the product of generalized force and the corresponding generalized displacement.
- Work done by a force is conservative if it is path independent.
- Non-linear systems and non-conservative systems are two independent description of a system.
- Conditions specified on deflections (and slopes) at the boundary are called Kinematic or Essential boundary conditions.
- Conditions specified on the internal forces and moments at the boundary are called Statical or Natural boundary conditions.
- Functions that are continuous and satisfies all the kinematic boundary conditions are called kinematically admissible functions.
- The virtual displacement is an infinitesimal imaginary kinematically admissible displacement field imposed on a body.
- Virtual work is the work done by the forces in moving through a virtual displacement.
- The total virtual work done on a body at equilibrium is zero.
Minimum Potential Energy

- Of all the kinematically admissible displacement functions the actual displacement function is the one that minimizes the potential energy function at stable equilibrium.

Potential energy function: \( \Omega = U - W \)

where, \( U \) is the strain energy and \( W \) is the work potential of a force.

Rayleigh-Ritz’s method

- Rayleigh-Ritz method is formal procedure to approximately minimize potential energy function.

\[
u(x) = \sum_{i=1}^{n} C_i f_i(x)
\]

where,

- \( C_i \) are undetermined constants (generalized displacement)
- \( f_i(x) \) are kinematically admissible functions that are independent and form a complete set.

The necessary condition at the minimum value of \( \Omega \) is:

\[
\frac{\partial \Omega}{\partial C_i} = 0 \quad i = 1, n
\]

\[
\sum_{j=1}^{n} K_{ij} C_j = R_i \quad i = 1, n \quad [K] \{C\} = \{R\} \quad K_{ji} = K_{ij}
\]

\[
U_A = \frac{W_A}{2} = \left( \frac{\Omega_A}{2} \right) = \frac{1}{2} \sum_{j=1}^{n} C_j R_j \quad \text{at equilibrium}
\]

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<td>( v(x) = \sum_{i=1}^{n} C_i f_i(x) )</td>
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<td>( K_{jk} = \int_0^L E A \left( \frac{d^2 f_j}{dx^2} \right) \left( \frac{d^2 f_k}{dx^2} \right) dx )</td>
<td>( K_{jk} = \int_0^L G J \left( \frac{d^2 f_j}{dx^2} \right) \left( \frac{d^2 f_k}{dx^2} \right) dx )</td>
<td>( K_{jk} = \int_0^L (E I_{zz}) \left( \frac{d^2 f_j}{dx^2} \right)^2 \left( \frac{d^2 f_k}{dx^2} \right)^2 dx )</td>
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<td>( R_j = \int_0^L p_x(x) f_j(x) ) dx</td>
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<td>( R_j = \int_0^L p_y(x) f_j(x) ) dx</td>
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<tr>
<td>( + \sum_{q=1}^{m} F_q f_j(x_q) )</td>
<td>( + \sum_{q=1}^{m} T_q f_j(x_q) )</td>
<td>( + \sum_{q=1}^{m} F_q f_j(x_q) + \sum_{q=1}^{m} M_q \frac{df_j}{dx} (x_q) )</td>
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**Finite Element Method**

- The kinematically admissible displacement functions in finite element method is defined piecewise continuous over small (finite) domains called the ‘elements’.
- The constants multiplying the piecewise kinematically admissible functions are the displacements of the nodes.
- The kinematically admissible functions are called ‘interpolation functions’ as these functions can be used to interpolate the values of displacements between the nodes.

\[
\begin{align*}
    u(x) &= \sum_{i=1}^{n} u_i^{(e)} f_i(x) \\
    \delta \Omega &= \sum_{e=1}^{n} \delta \Omega^{(e)} \\
    [K_G^{(e)}] &= [T]^T[K^{(e)}][T] \\
    \{F_{G1}^{(e)}\} &= [T]^T\{F_{1}^{(e)}\} \\
    \delta \Omega^{(e)} &= \{\delta u_G^{(e)}\}^T \left( [K_G^{(e)}]\{u_G^{(e)}\} - \{F_{G1}^{(e)}\} \right)
\end{align*}
\]

**Function Continuity (Lagrange Polynomials)**

Displacement Continuity: Axial, Torsion, 2-D and 3-D Elasticity
Temperature Continuity: Heat Transfer 1-D, 2-D, and 3-D

Langrange Polynomials:
\[
    f_i(x_j) = \begin{cases} 
        1 & i = j \\
        0 & i \neq j 
    \end{cases}
\]

**Function and Derivative Continuity (Hermite Polynomials)**

Displacement and Slope Continuity: Beams, Plates, Shells.

**Steps in FEM procedure**

1. Obtain element stiffness and element load vector.
2. Transform from local orientation to global orientation.
3. Assemble the global stiffness matrix and load vector.
4. Incorporate the external loads
5. Incorporate the boundary conditions.
6. Solve the algebraic equations for nodal displacements.
7. Obtain reaction force, stress, internal forces, strain energy.
8. Interpret and check the results.
9. Refine mesh if necessary, and repeat the above steps.

**Natural Coordinates**

- Coordinates which vary between 0 and 1 or -1 and 1.
- Natural coordinates and non-dimensional coordinates.
1-d Coordinates

Possibility 1

\[ L_J(\xi) = (1 - \xi) \]
\[ L_2(\xi) = \xi \]

Possibility 2

\[ L_J(\xi) = (1 - \xi) / 2 \]
\[ L_2(\xi) = (1 + \xi) / 2 \]

2-D Triangular elements (Area Coordinates)

- Use Pascal’s triangle to determine the nodes needed for complete polynomials

\[
L_I = \frac{A_I}{A} \quad L_J = \frac{A_J}{A} \\
L_K = \frac{A_K}{A} \quad L_I + L_J + L_K = 1
\]

Bi-linear

Tri-linear

Numerical Integration (Gauss Quadrature)

\[
1 = \int_{-1}^{1} F(\xi) d\xi = \sum_{i=1}^{n} w_i F(\xi_i) \\
1 = \int_{-1}^{1} \int_{-1}^{1} F(\xi, \eta)(d\xi)d\eta = \sum_{j=1}^{n} \sum_{i=1}^{n} w_i w_j F(\xi_i, \eta_j) \\
1 = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} F(\xi, \eta, \zeta)(d\xi)d\eta = \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} w_i w_j w_k F(\xi_i, \eta_j, \zeta_k)
\]

- Stresses are found at Gauss points.
Iso-Parametric Elements

- Displacements and coordinates are approximated by same interpolation functions.

\[
\begin{align*}
    u &= \sum_{i=1}^{n} N_i(\xi, \eta) u_i^{(e)} \\
    v &= \sum_{i=1}^{n} N_i(\xi, \eta) v_i^{(e)} \\
    x &= \sum_{i=1}^{n} N_i(\xi, \eta) x_i \\
    y &= \sum_{i=1}^{n} N_i(\xi, \eta) y_i
\end{align*}
\]

Jacobian Matrix

- Matrix relating differentials of two coordinate system.

\[
\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \xi} \\
\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \eta}
\]

\[
[J] = \begin{bmatrix}
    \sum_{i=1}^{m} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{m} \frac{\partial N_i}{\partial \eta} y_i \\
    \sum_{j=1}^{m} \frac{\partial N_j}{\partial \xi} x_i & \sum_{i=1}^{m} \frac{\partial N_i}{\partial \eta} y_i
\end{bmatrix}
\]

\[
K^{(e)} = \int\int [B]^T [E] [B] |J| (dx)(dy) = \int\int [\tilde{B}]^T [E] [\tilde{B}] |J| (d\xi)(d\eta)
\]

Storage and Solution Techniques

1. Banded Matrix
   - The bandwidth is solely dictated by the difference between the highest and the lowest node number on the elements.

2. Skyline storage scheme
   - The column above the diagonal up to the last non-zero is stored as a vector.

3. Wavefront (Frontal) equation solver.
   - The coefficient in the matrix is processed as soon as the contribution of all elements is complete. Wavefront number is a measure of number of coefficients being

4. Substructuring (Matrix partitioning)
   - The structure is sub-structured and the degrees of freedom (DOF) of the substructure boundary nodes are first found and then the DOF of the internal nodes in each substructure are found separately.
Errors in FEM

1. Modeling error
   • Errors in creating the boundary value problem to represent the real structure.

2. Discretization error
   • Errors that arises from creation of the mesh.

3. Numerical error
   • Errors that arise from finite digit arithmetic and use of numerical methods.

FEM Convergence

1. Does the FEM solution converge as mesh is refined?
2. Does the FEM solution converge rapidly?
3. Does the FEM solution converge to the right solution?

Patch Test

• A numerical test designed to check if a mesh made from a specific type of element will converge as it is refined.

Mesh Refinement

• Elements with high strain energy identify the region of the body where mesh should be refined.

• Regions showing large jumps of stresses across the element boundary should be refined,

1. The h-method of mesh refinement reduces the size of element.
2. The p-method of mesh refinement increases the order of polynomial in an element.
3. The r-method of mesh refinement relocates the position of a node.
4. Combinations: hr-method, hp-method, hpr-method
Thermal Analysis

Heat Conduction

\[
\frac{d}{dx}\left(k \frac{dT}{dx}\right) = -q_x
\]

k = Thermal conductivity; \( q_x \) = Heat flow (source) per unit length.; \( f_x \) = heat flux per unit area = \( q_x A \)

Heat flow: \( q = -k \frac{dT}{dx} \); Positive: Flow into body

- Heat conduction problem is analogous to axial problem.
  \( u \rightarrow T \quad N \rightarrow q \quad p_x \rightarrow q_x \quad EA \rightarrow k \)

Thermal Stresses

\[
\varepsilon_{xx} = \frac{du}{dx} = \frac{\sigma_{xx}}{E} + \alpha \Delta T
\]

- Assume that the heat conduction problem can be solved independently and before stress and deformation problem.
  \[
  \begin{cases}
    F_{T1} = -E A \alpha \Delta T_1 \\
    F_{T2} = E A \alpha \Delta T_2
  \end{cases}
  \quad p_x T = \frac{E A \alpha}{k} q_x
\]

2-D Steady State Thermal Analysis

- Differential Equation:
  \[
  k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) = -q_v
  \text{ or } k \nabla^2 T = -q_v
\]

- Functional (Stored Heat):
  \[
  U_T = \int_A \int \left[ k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)^2 \right] \text{d}x \text{d}y
  \]

- Element Approximation:
  \[
  T(x) = \sum_{i=1}^{n} T_i^{(e)} f_i(x, y)
  \]
  \( f_i(x, y) \) are Lagrange Polynomials.

\[
[B^{(e)}] = \begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial x} & \cdots & \frac{\partial f_n}{\partial x} \\
\frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial y} & \cdots & \frac{\partial f_n}{\partial y}
\end{bmatrix}
\]

- Element Conductivity Matrix:
  \[
  [K_T^{(e)}] = \int_A [B^{(e)}]^T [k^{(e)}] [B^{(e)}] \text{d}x \text{d}y
  \]
  Isotropic material: \( k^{(e)} = k^{(e)} \) --- scaler quantity.
Heat Conduction Boundary Conditions: 
\[-k \frac{\partial T}{\partial n} = -k \left( \frac{\partial T}{\partial x} n_x + \frac{\partial T}{\partial y} n_y \right) = q_n\]

- Right Hand Side Vector: \( \{R^{(e)}\} = \int_{\Lambda^{(e)}} \int q_n \, t \, dx \, dy + \int_{\Gamma^{(e)}} q_n \, t \, ds \)

Convection Boundary Conditions: 
\[-k \frac{\partial T}{\partial n} = h(T_f - T)\]

- Addition to Element Matrix: 
  \[K^{(e)}_{ij} = \int_{\Gamma^{(e)}} h^{(e)} f_i f_j t \, ds\]

- Addition to Element Right Hand Side Vector: 
  \[R^{(e)}_i = \int_{\Gamma^{(e)}} h^{(e)} f_i T_f t \, ds\]

If \(\Gamma^{(e)}\) is an element boundary in the interior, then there is no convection there and hence no addition to the matrix or the element RHS vector.

Radiation Boundary Condition:
\[-k \frac{\partial T}{\partial n} = B_C(T_f^4 - T^4)\]

- Radiation boundary conditions lead to non-linear thermal problem.

A general approach is:
\[-k \frac{\partial T}{\partial n} = B_C(T_f^2 + T^2)(T_f + T)(T_f - T) = h_r(T_f - T)\]

where, \(B_C\) is the proportionality constant, \(T_f\) is the temperature of the other radiating body, Temperatures \(T_f\) and \(T\) are in absolute degrees i.e., \(^0\)K.

FEM is a very versatile and powerful analysis tool. Proper use of this numerical tool to produce correct results is the responsibility of the user.