MEEM 4150 REVIEW

Stress at a Point:

$$\sigma_{ij} = \lim_{\Delta A_i \to 0}\left(\frac{\Delta F_j}{\Delta A_i}\right)$$

direction of outward normal to the imaginary cut surface (i). direction of the internal force (j).

1. Stress is an internal quantity.
2. Stress has units of force per unit area.
3. Stress at a point needs a magnitude and two directions to specify it (i.e. stress is a second-order tensor).
4. The sign of a stress component is determined from the direction of the internal force and the direction of the outward normal to the imaginary cut surface.

Stress Transformation:

$$\sigma_{nn} = \{n\}^T[\sigma]\{n\} \qquad \tau_{nt} = \{t\}^T[\sigma]\{n\} \qquad \sigma_{tt} = \{t\}^T[\sigma]\{t\} \qquad \{S\} = [\sigma]\{n\} \qquad \sigma_{nn}\{n\} + \tau_{nt}\{t\}$$

Principal Stresses:

1. The eigenvalues of the stress matrix are the principal stresses.
2. The eigenvectors of the stress matrix are the principal directions.

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0 \qquad I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \qquad I_2 = \begin{vmatrix}\sigma_{xx} & \tau_{xy}\\ \tau_{yx} & \sigma_{yy}\end{vmatrix} + \begin{vmatrix}\sigma_{yy} & \tau_{yz}\\ \tau_{zy} & \sigma_{zz}\end{vmatrix} + \begin{vmatrix}\sigma_{xx} & \tau_{xz}\\ \tau_{zx} & \sigma_{zz}\end{vmatrix} \qquad I_3 = \begin{vmatrix}\sigma_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_{zz}\end{vmatrix}$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \qquad I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \qquad I_3 = \sigma_1\sigma_2\sigma_3$$

The roots of the equation $x^3 - I_1x^2 + I_2x - I_3 = 0$ are $x_1 = 2A\cos\alpha + I_1/3$,and $x_{2,3} = -2A\cos(\alpha \pm 60^o) + I_1/3$, where $A = \sqrt{(I_1/3)^2 - I_2/3}$ and

$\cos 3\alpha = [2(I_1/3)^3 - (I_1/3)I_2 + I_3]/(2A^3)$

Strain:

Measure of relative movement of two points on the body. (deformation)

Strain at a point:

1. Needs magnitude and two direction to specify it.
2. Is related to the first partial derivative of deformation.
3. Strain is a symmetric. In 3-D: 6 components are needed to specify strain at a point. In 2-D: 3 components are needed to specify strain at a point.
4. Elongations are positive normal strains. Decrease from right angle results in positive shear strains.
5. Normal small strain ($\varepsilon < 0.01$) can be calculated using just the deformation in the original direction of the line.

Engineering Strain $\varepsilon_{xx} = \frac{\partial u}{\partial x}$ $\quad \varepsilon_{yy} = \frac{\partial v}{\partial y}$ $\quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$ $\quad \gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ $\quad \gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$ $\quad \gamma_{zx} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$

6. tensor normal strains = engineering normal strains and tensor shear strains = (engineering shear strains)/ 2

Finite Difference: Forward Difference: $(\varepsilon_{xx})_i = \frac{u_{i+1} - u_i}{x_{i+1} - x_i}$. Backward difference $(\varepsilon_{xx})_i = \frac{u_i - u_{i-1}}{x_i - x_{i-1}}$. Central difference $(\varepsilon_{xx})_i = \frac{1}{2}\left[\frac{u_{i+1} - u_i}{x_{i+1} - x_i} + \frac{u_i - u_{i-1}}{x_i - x_{i-1}}\right]$

Strain Transformation: Convert Engineering strain to tensor strain and use the formulas for stress transformation

Generalized Hooke's Law:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz}) + \alpha\Delta T \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}) + \alpha\Delta T \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) + \alpha\Delta T \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

$$G = \frac{E}{2(1+\nu)}$$

Assuming no temperature change, we have the following:

Plane Stress → $\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ → Generalized Hooke's Law → $\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$

Plane Strain → $\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ → Generalized Hooke's Law → $\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$

Stress Concentration Factor: $K_{conc} = \frac{Maximum\ \ Stress}{Nominal\ \ Stress}$.

Stress Intensity Factor:

- Stress intensity factor depends upon the stress level and the length of the crack.
- Critical stress intensity factor is a material property that is independent of the stress level or crack length.
- A crack becomes unstable (material breaks) when stress intensity factor exceeds the critical stress intensity factor.
- Microcracks will be assumed to grow in Mode I due to principal stress one if it is in tension.

$$K_I = \sigma_{Noninal}\sqrt{\pi a} \qquad K_{II} = \tau_{Nominal}\sqrt{\pi a} \qquad K_{equivalent} = \sqrt{K_I^2 + K_{II}^2}$$

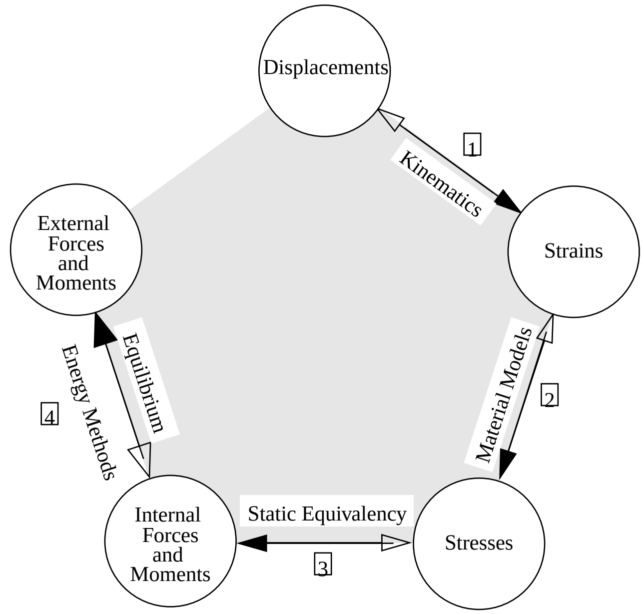
Maximum Shear Stress Theory: $|max(\sigma_1 - \sigma_2, \sigma_2 - \sigma_3, \sigma_3 - \sigma_1)| \leq \sigma_{yield}$ ---ductile material

Maximum Octahedral Shear Stress Theory: $\sigma_{von} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \leq \sigma_{yield}$ ---ductile material

Maximum Normal Stress Theory $|max(\sigma_1, \sigma_2, \sigma_3)| \leq \sigma_{ult}$ ----brittle material

Modified Mohr's Theory: $\left|\frac{\sigma_2}{\sigma_C} - \frac{\sigma_1}{\sigma_T}\right| \leq 1$ ---brittle material

Structural Mechanics



	Axial (Rods)	Torsion (Shafts)	Symmetric Bending (Beams)	Unsymmetric Bending
Displace-ments	$u(x, y, z) = u(x)$	$\phi(x, y, z) = \phi(x)$	$u(x, y, z) = -y\frac{dv}{dx}$ $v = v(x)$ $w = 0$	$u(x, y, z) = -y\frac{dv}{dx} - z\frac{dw}{dx}$ $v = v(x)$ $w = w(x)$
Strains	$\varepsilon_{xx} = \frac{du}{dx}$	$\gamma_{x\theta} = \rho\frac{d\phi}{dx}$	$\varepsilon_{xx} = -y\frac{d^2v}{dx^2}$	$\varepsilon_{xx} = -y\frac{d^2v}{dx^2} - z\frac{d^2w}{dx^2}$
Stresses	$\sigma_{xx} = E\varepsilon_{xx} = E\frac{du}{dx}$	$\tau_{x\theta} = G\gamma_{x\theta} = G\rho\frac{d\phi}{dx}$	$\sigma_{xx} = E\varepsilon_{xx} = -Ey\frac{d^2v}{dx^2}$ $\tau_{xy} \neq 0 \ll \sigma_{xx}$	$\sigma_{xx} = -Ey\frac{d^2v}{dx^2} - Ez\frac{d^2w}{dx^2}$ $\tau_{xy} \neq 0 \ll \sigma_{xx}$ $\tau_{xz} \neq 0 \ll \sigma_{xx}$
Internal Forces & Moments	$\boldsymbol{N} = \int_A \sigma_{xx} dA$	$\boldsymbol{T} = \int_A \rho\tau_{x\theta} dA$	$\boldsymbol{N} = \int_A \sigma_{xx} dA = 0$ $M_z = -\int_A y\sigma_{xx} dA$ $V_y = \int_A \tau_{xy} dA$	$\boldsymbol{N} = \int_A \sigma_{xx} dA = 0$ $\boldsymbol{M}_z = -\int_A y\sigma_{xx} dA$ $\boldsymbol{M}_y = -\int_A z\sigma_{xx} dA$ $\boldsymbol{V}_y = \int_A \tau_{xy} dA$ $\boldsymbol{V}_z = \int_A \tau_{xz} dA$

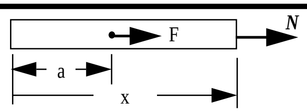
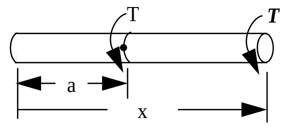
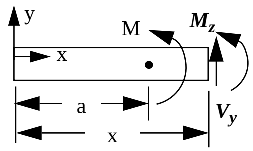
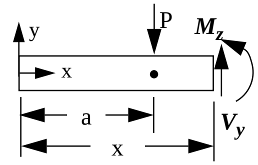
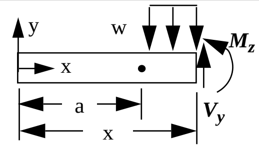
	Axial (Rods)	Torsion (Shafts)	Symmetric Bending (Beams)	Unsymmetric Bending
	$\sigma_{xx} = \frac{\boldsymbol{N}}{A}$	$\tau_{x\theta} = \frac{\boldsymbol{T}\rho}{J}$	$\sigma_{xx} = -\left(\frac{\boldsymbol{M}_z y}{I_{zz}}\right)$ $q = \tau_{xs} t = -\left(\frac{\boldsymbol{V}_y Q_z}{I_{zz}}\right)$	$\sigma_{xx} = -\left(\frac{I_{yy}\boldsymbol{M}_z - I_{yz}\boldsymbol{M}_y}{I_{yy}I_{zz} - I_{yz}^2}\right)y - \left(\frac{I_{zz}\boldsymbol{M}_y - I_{yz}\boldsymbol{M}_z}{I_{yy}I_{zz} - I_{yz}^2}\right)z$ $q = \tau_{xs} t = -\left(\frac{I_{yy}Q_z - I_{yz}Q_y}{I_{yy}I_{zz} - I_{yz}^2}\right)\boldsymbol{V}_y - \left(\frac{I_{zz}Q_y - I_{yz}Q_z}{I_{yy}I_{zz} - I_{yz}^2}\right)\boldsymbol{V}_z$
	$\frac{du}{dx} = \frac{\boldsymbol{N}}{EA}$ $u_2 - u_1 = \frac{\boldsymbol{N}(x_2 - x_1)}{EA}$	$\frac{d\phi}{dx} = \frac{\boldsymbol{T}}{GJ}$ $\phi_2 - \phi_1 = \frac{\boldsymbol{T}(x_2 - x_1)}{GJ}$	$\frac{d^2 v}{dx^2} = \frac{\boldsymbol{M}_z}{EI_{zz}}$ $v = \int\left[\int \frac{\boldsymbol{M}_z}{EI} dx\right] dx + C_1 x + C_2$	$\frac{d^2 v}{dx^2} = \frac{1}{E}\left(\frac{I_{yy}\boldsymbol{M}_z - I_{yz}\boldsymbol{M}_y}{I_{yy}I_{zz} - I_{yz}^2}\right)$ $\quad \frac{d^2 w}{dx^2} = \frac{1}{E}\left(\frac{I_{zz}\boldsymbol{M}_y - I_{yz}\boldsymbol{M}_z}{I_{yy}I_{zz} - I_{yz}^2}\right)$
	$(\sigma_{xx})_i = \frac{\boldsymbol{N}E_i}{\sum_{j=1}^{n} E_j A_j}$	$(\tau_{x\theta})_i = \frac{G_i \rho \boldsymbol{T}}{\left[\sum_{j=1}^{n} G_j J_j\right]}$	$(\sigma_{xx})_i = -\left\lvert\frac{E_i y \boldsymbol{M}_z}{\left[\sum_{j=1}^{n} E_j (I_{zz})_j\right]}\right\rvert$ $\quad q = \tau_{xs} t = -\left\{\frac{Q_{comp}\boldsymbol{V}_y}{\left[\sum_{j=1}^{n} E_j (I_{zz})_j\right]}\right\}$	
	$u_2 - u_1 = \frac{\boldsymbol{N}(x_2 - x_1)}{\sum E_j A_j}$	$\phi_2 - \phi_1 = \frac{\boldsymbol{T}(x_2 - x_1)}{[\sum G_j J_j]}$	$v = \int\left[\int \frac{\boldsymbol{M}_z}{\sum E_j (I_{zz})_j} dx\right] dx + C_1 x + C_2$	
	$\frac{d\boldsymbol{N}}{dx} = -p_x(x)$	$\frac{d\boldsymbol{T}}{dx} = -t(x)$	$\frac{d\boldsymbol{V}_y}{dx} = -p_y(x)$ $\quad \frac{d\boldsymbol{M}_z}{dx} = -\boldsymbol{V}_y$	$\frac{d\boldsymbol{V}_y}{dx} = -p_y(x)$ $\quad \frac{d\boldsymbol{M}_z}{dx} = -\boldsymbol{V}_y$ $\quad \frac{d\boldsymbol{V}_z}{dx} = -p_z(x)$ $\frac{d\boldsymbol{M}_y}{dx} = -\boldsymbol{V}_z$
	$\frac{d}{dx}\left(EA\frac{du_o}{dx}\right) = -p_x(x)$	$\frac{d}{dx}\left(GJ\frac{d\phi}{dx}\right) = -t(x)$	$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2 v}{dx^2}\right) = p_y(x)$	

Discontinuity Functions

$$\langle x - a \rangle^n = \begin{Bmatrix} 0 & x \le a \\ (x - a)^n & x > a \end{Bmatrix}$$

$$\int_{-\infty}^{x} \langle x - a \rangle^{-2} dx = \langle x - a \rangle^{-1} \qquad \int_{-\infty}^{x} \langle x - a \rangle^{-1} dx = \langle x - a \rangle^{-0} \qquad \int_{-\infty}^{x} \langle x - a \rangle^{n} dx = \frac{\langle x - a \rangle^{n+1}}{(n+1)} \qquad n \ge 0$$

$$\frac{d\langle x-a\rangle^{-1}}{dx} = \langle x-a\rangle^{-2} \qquad \frac{d\langle x-a\rangle^{0}}{dx} = \langle x-a\rangle^{-1} \qquad \frac{d\langle x-a\rangle^{n}}{dx} = n\langle x-a\rangle^{n-1} \qquad n \geq 1$$

	Axial (Rods)	Torsion (Shafts)	Bending (Beams)		
Templates					
Equations	$\boldsymbol{N} = -F\langle x-a\rangle^{0}$ $p_x = F\langle x-a\rangle^{-1}$	$\boldsymbol{T} = -T\langle x-a\rangle^{0}$ $t = T\langle x-a\rangle^{-1}$	$\boldsymbol{M_z} = -M\langle x-a\rangle^{0}$ $p_y = -M\langle x-a\rangle^{-2}$	$\boldsymbol{M_z} = -P\langle x-a\rangle^{1}$ $p_y = -P\langle x-a\rangle^{-1}$	$\boldsymbol{M_z} = -w\frac{\langle x-a\rangle^{2}}{2}$ $p_y = -w\langle x-a\rangle^{0}$

Shear Center

- Shear center is a point in space at which the shear stress due to bending can be replaced by statically equivalent internal shear forces and no internal torque.

or

- Shear center is a point in space such that if the line of action of external forces pass through the point then the cross-section will not twist.

Each cross-section has a unique shear center associated with it.
Shear center depends only on the geometry and is independent of the loading.
Shear center lies on the axis around which the shear stress distribution is symmetric.
Shear center de-couples the shear stresses due to bending from the shear stresses due to torsion.

Thin Closed Section

$$q_c = q_o + q \qquad q = \tau_{sx}t = -\left(\frac{I_{yy}Q_z - I_{yz}Q_y}{I_{yy}I_{zz} - I_{yz}^2}\right)V_y - \left(\frac{I_{zz}Q_y - I_{yz}Q_z}{I_{yy}I_{zz} - I_{yz}^2}\right)V_z \qquad \oint\left(\frac{q_c}{t}\right)ds = \oint\left(\frac{q_o + q}{t}\right)ds = 0$$

Material Models

- Linear, Elastic, Homogenous, Isotropic Material with no temperature change. $\sigma_{xx} = E\varepsilon_{xx}$ Origin: centroid of cross-section
- Linear, Elastic, Homogenous, Isotropic Material with temperature change. $\sigma_{xx} = E(\varepsilon_{xx} - \alpha\Delta T)$ Origin: centroid of cross-section
- Linear, Elastic, Non-homogenous (Composite), Isotropic Material. $(\sigma_{xx})_i = E_i\varepsilon_{xx}$ Origin: $\eta_c = \left(\sum_{i=1}^{n}\eta_i A_i\right)\Big/\left(\sum_{i=1}^{n}A_i\right)$

- Non-linear Material Models Origin: $\boldsymbol{N} = \int_A \sigma_{xx} dA = 0$

(i) Elastic-perfectly plastic in which the non-linearity is approximated by a constant.

$$\sigma = \begin{cases} \sigma_{yield} & \varepsilon \geq \varepsilon_{yield} \\ E\varepsilon & -\varepsilon_{yield} \leq \varepsilon \leq \varepsilon_{yield} \\ -\sigma_{yield} & \varepsilon \leq -\varepsilon_{yield} \end{cases} \qquad \tau = \begin{cases} \tau_{yield} & \gamma \geq \gamma_{yield} \\ G\gamma & -\gamma_{yield} \leq \gamma \leq \gamma_{yield} \\ -\tau_{yield} & \gamma \leq -\gamma_{yield} \end{cases}$$

(ii) Linear strain hardening model in which the non-linearity is approximated by a linear function.

$$\sigma = \begin{cases} \sigma_{yield} + E_2(\varepsilon - \varepsilon_{yield}) & \varepsilon \geq \varepsilon_{yield} \\ E_1\varepsilon & -\varepsilon_{yield} \leq \varepsilon \leq \varepsilon_{yield} \\ -\sigma_{yield} + E_2(\varepsilon + \varepsilon_{yield}) & \varepsilon \geq \varepsilon_{yield} \end{cases} \qquad \tau = \begin{cases} \tau_{yield} + G_2(\gamma - \gamma_{yield}) & \gamma \geq \gamma_{yield} \\ G_1\gamma & -\gamma_{yield} \leq \gamma \leq \gamma_{yield} \\ -\tau_{yield} + G_2(\gamma + \gamma_{yield}) & \gamma \leq \gamma_{yield} \end{cases}$$

(iii) Power law model in which the non-linearity is approximated by one term non-linear function.

$$\sigma = \begin{cases} E\varepsilon^n & \varepsilon \geq 0 \\ -E(-\varepsilon)^n & \varepsilon < 0 \end{cases} \qquad \tau = \begin{cases} G\gamma^n & \gamma \geq 0 \\ -G(-\gamma)^n & \gamma < 0 \end{cases}$$

1.The set of points forming the boundary between the elastic and plastic region on a body, is called the elastic-plastic boundary.
2.On the elastic-plastic boundary the strain must be equal to the yield strain, and stress equal to yield stress.

Energy Methods

- The strain energy per unit volume is called the *strain energy density* and is the area underneath the stress-strain curve up to the point of deformation. $U_o = \int_0^{\varepsilon} \sigma d\varepsilon$
- The energy stored in a body due to deformation is called the *strain energy.* $U = \int_V U_o \; dV$
- Complimentary strain energy density. $\bar{U}_o = \int_0^{\sigma} \varepsilon d\sigma$
- The strain energy density at the yield point is called *Modulus of Resilience.*
- The strain energy density at rupture is called *Modulus of Toughness*.

Linear Strain Energy Density: $U_o = \frac{1}{2}\sigma\varepsilon \qquad U_o = \frac{1}{2}[\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}]$

	Axial (Rods)	Torsion (Shafts)	Bending (Beams)
Strain Energy	$U_a = \frac{1}{2}EA\left(\frac{du}{dx}\right)^2$	$U_t = \frac{1}{2}GJ\left(\frac{d\phi}{dx}\right)^2$	$U_b = \frac{1}{2}EI_{zz}\left(\frac{d^2v}{dx^2}\right)^2$
Complimentary Strain Energy	$\overline{U}_a = \frac{1}{2}\frac{\boldsymbol{N}^2}{EA}$	$\overline{U}_t = \frac{1}{2}\frac{\boldsymbol{T}^2}{GJ}$	$\overline{U}_b = \frac{1}{2}\frac{\boldsymbol{M}_z^2}{EI_{zz}}$

- Any variable that can be used for describing deformation is called the generalized displacement.
- Any variable that can be used for describing the cause that produces deformation is called the generalized force.

Virtual work theorem: The total virtual work done on a body at equilibrium is zero. $\delta W_{ext} = \delta W_{int}$

- Functions that are continuous and satisfies all the kinematic boundary conditions are called kinematically admissible functions.
- Functions that satisfy satisfies all the static boundary conditions, satisfy equilibrium equations at all points, are continuous at all points except where a concentrated force or moment is applied are called statically admissible functions.
- In determining statically admissible internal forces and moments, the number of reactions that can be assigned arbitrary values is equal to the degree of redundancy.
- The virtual displacement is an infinitesimal imaginary kinematically admissible displacement field imposed on a body.
- Virtual work is the work done by the forces in moving through a virtual displacement.
- The virtual force is an infinitesimal imaginary statically admissible force field imposed on a body.

Dummy Unit Load Method : $(F = 1)v_1(x_P) = \int_0^L \frac{\boldsymbol{M}_2(x)\boldsymbol{M}_1(x)}{EI}dx$ $\qquad (M = 1)\frac{dv_1}{dx}(x_P) = \int_0^L \frac{\boldsymbol{M}_2(x)\boldsymbol{M}_1(x)}{EI}dx$

- A positive sign for v_1 implies that the deflection is in the same direction as the applied unit force. A positive sign for $\frac{dv_1}{dx}$ implies that the slope is in the same direction as the applied unit moment.

Castigliano's Method $v_1(x_P) = \frac{\partial \overline{U}_B}{\partial F}$ $\qquad \frac{dv_1}{dx}(x_P) = \frac{\partial \overline{U}_B}{\partial M}$

- A positive sign for v_1 implies that the deflection is in the same direction as the applied force. A positive sign for $\frac{dv_1}{dx}$ implies that in the same direction as the applied moment.