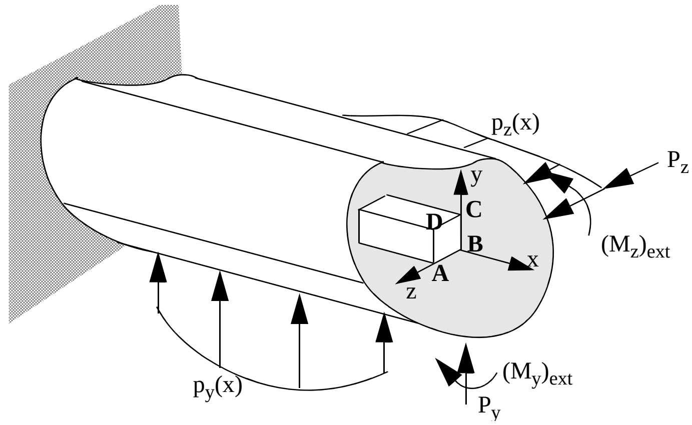
Unsymmetric Bending of Beams

- Drop the limitation that the beam has a plane of symmetry and the loading is in the plane of symmetry.
- Assume loading is such that there is no twisting of the cross-section.

The learning objectives of this chapter are:

- Understand the theory, its limitations, and its application in design and analysis of unsymmetric bending of beam.
- Understand the concept of shear center and how to determine its location.

Theory

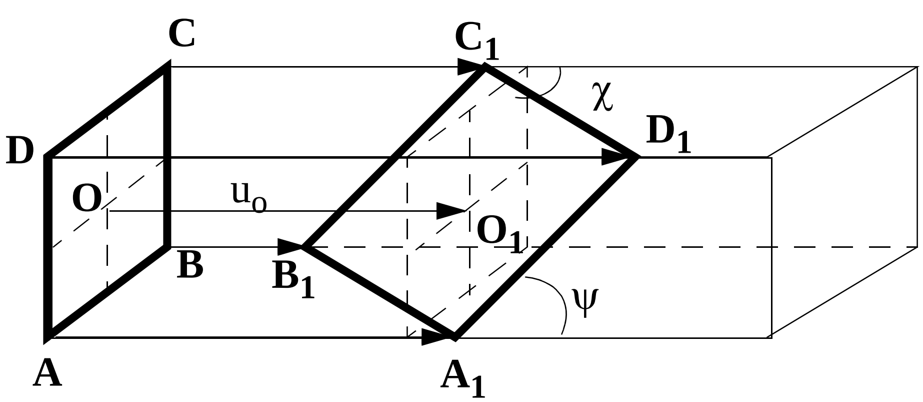


Theory objective is:

- Relate the internal shear forces V_y, V_z and internal moment M_y, M_z to displacements v and w and obtain the stresses in unsymmetric bending.

Deformation Behavior

Assumption 1 The loads are such that there is no axial or torsional deformation.



No twist implies: $\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$

$$v(x, y, z) = v(x, y) \qquad w(x, y, z) = w(x, z)$$

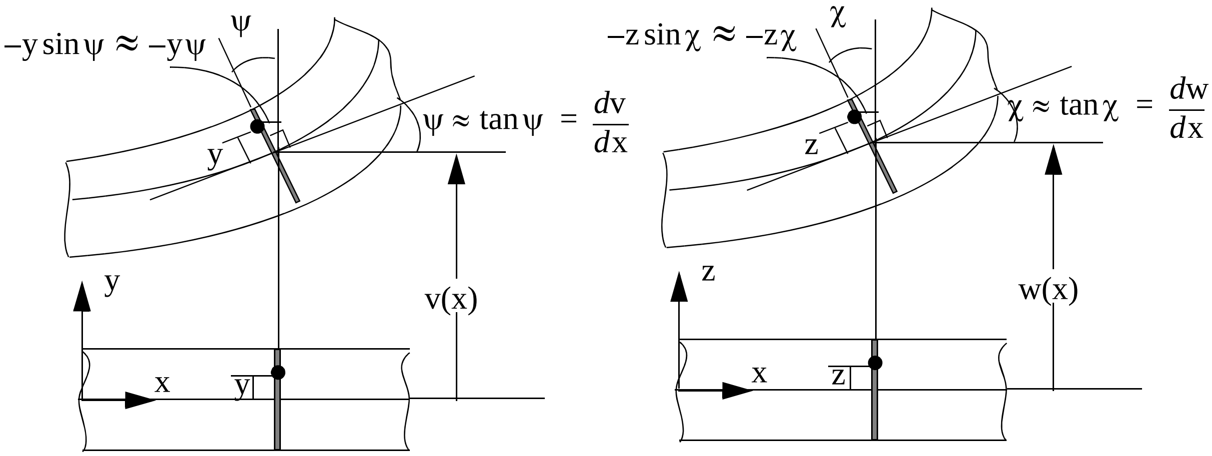
Assumption 2 Squashing action is significantly smaller then bending action.

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \approx 0 \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} \approx 0$$

$$v = v(x) \qquad w = w(x)$$

Assumption 3 Plane sections before deformation remain plane after deformation.

$$u = u_o - \psi y - \chi z$$



Assumption 4 Plane perpendicular to the axis remain nearly perpendicular after deformation.

$$u = -y\frac{dv}{dx} - z\frac{dw}{dx}$$

Strain Distribution

Assumption 5 Strains are small.

$$\varepsilon_{xx} = \frac{du}{dx} = -y\frac{d^2v}{dx^2} - z\frac{d^2w}{dx^2}$$

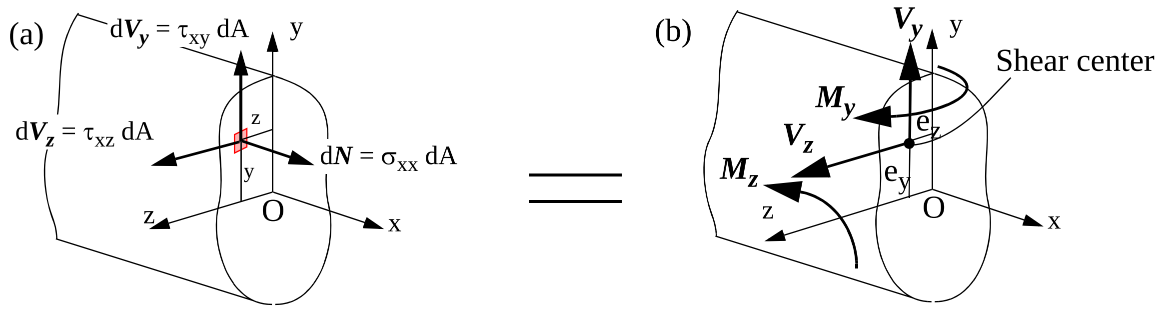
Material Model

Assumption 6 Material is isotropic
Assumption 7 Material is elastic.
Assumption 8 Stress and strains are linearly related
Assumption 9 There are no inelastic strain.

Hooke's Law: $\sigma_{xx} = E\varepsilon_{xx}$

$$\sigma_{xx} = -Ey\frac{d^2v}{dx^2} - Ez\frac{d^2w}{dx^2}$$

Internal forces and moments



$$N = \int_A \sigma_{xx} dA = 0$$

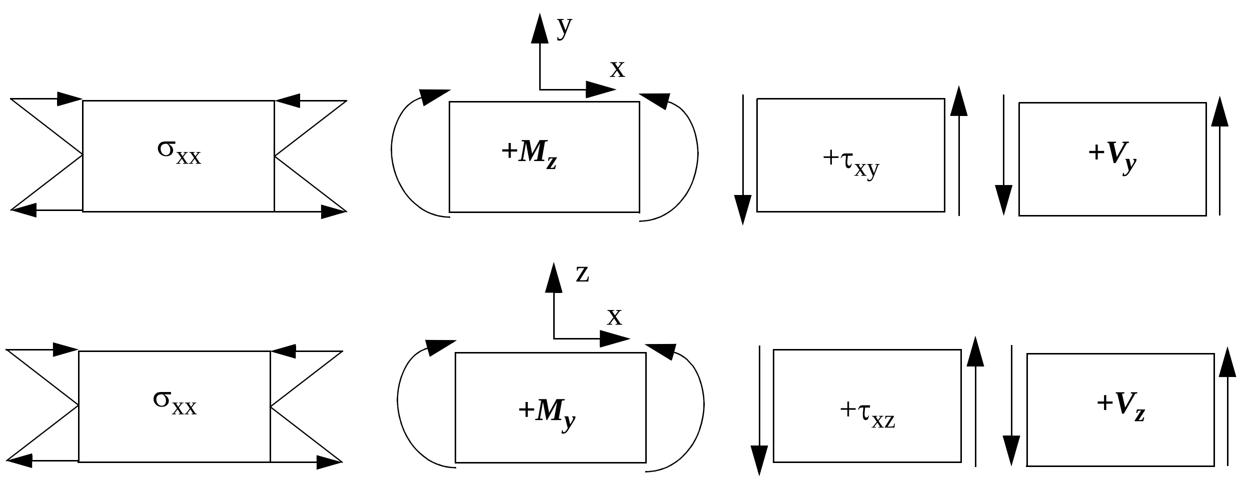
$$M_z = -\int_A y\sigma_{xx} dA \qquad M_y = -\int_A z\sigma_{xx} dA$$

$$V_y = \int_A \tau_{xy} dA \qquad V_z = \int_A \tau_{xz} dA$$

$$T = \int_A [(y - e_y)\tau_{xz} - (z - e_z)\tau_{xy}] dA = 0$$

- The maximum normal stress σ_{xx} in the beam should be nearly an order of magnitude (factor of 10) greater than the maximum shear stress τ_{xy} and τ_{xz}.

Sign Convention



Bending Formulas

Substituting $\sigma_{xx} = -Ey\frac{d^2v}{dx^2} - Ez\frac{d^2w}{dx^2}$ into internal moment expression.

$$\boldsymbol{M_z} = \frac{d^2v}{dx^2}\int_A Ey^2 dA + \frac{d^2w}{dx^2}\int_A EyzdA \qquad \boldsymbol{M_y} = \frac{d^2v}{dx^2}\int_A EyzdA + \frac{d^2w}{dx^2}\int_A Ez^2 dA$$

Assumption 10 Material is homogenous across the cross-section.

$$\boldsymbol{M_z} = EI_{zz}\frac{d^2v}{dx^2} + EI_{yz}\frac{d^2w}{dx^2} \qquad \boldsymbol{M_y} = EI_{yz}\frac{d^2v}{dx^2} + EI_{yy}\frac{d^2w}{dx^2}$$

Area moment of inertia

$$I_{zz} = \int_A y^2 dA \qquad I_{zz} = \int_A y^2 dA \qquad I_{yz} = \int_A yzdA$$

Moment Curvature Relationship

$$\frac{d^2v}{dx^2} = \frac{1}{E}\left(\frac{I_{yy}\boldsymbol{M_z} - I_{yz}\boldsymbol{M_y}}{I_{yy}I_{zz} - I_{yz}^2}\right) \qquad \frac{d^2w}{dx^2} = \frac{1}{E}\left(\frac{I_{zz}\boldsymbol{M_y} - I_{yz}\boldsymbol{M_z}}{I_{yy}I_{zz} - I_{yz}^2}\right)$$

Stress Formula

$$\sigma_{xx} = -\left(\frac{I_{yy}\boldsymbol{M_z} - I_{yz}\boldsymbol{M_y}}{I_{yy}I_{zz} - I_{yz}^2}\right)y - \left(\frac{I_{zz}\boldsymbol{M_y} - I_{yz}\boldsymbol{M_z}}{I_{yy}I_{zz} - I_{yz}^2}\right)z$$

Location of origin

Centroid: $\int_A ydA = 0$ $\int_A zdA = 0$

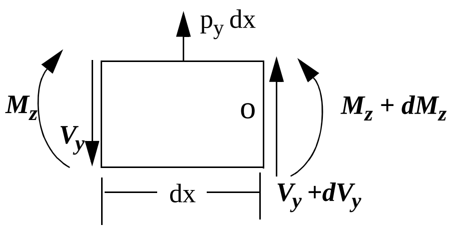
- The origin of the coordinate system must be the centroid of a homogenous cross-section
- Normal stress σ_{xx} in bending varies linearly with y and z on a homogenous cross-section.

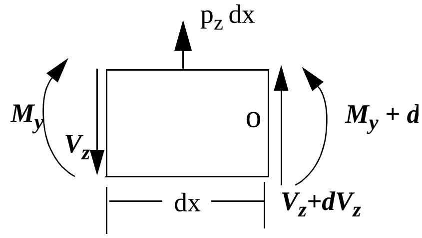
Neutral Axis ($\sigma_{xx} = 0$)

N.A. equation: $y = (\tan\beta)z$ $\tan\beta = \dfrac{I_{zz} - I_{yz}(\boldsymbol{M_z}/\boldsymbol{M_y})}{I_{yz} - I_{yy}(\boldsymbol{M_z}/\boldsymbol{M_y})}$

- The orientation of the neutral axis depends upon the shape of cross-section as well as the external loading.
- Bending normal stress σ_{xx} is maximum at the point which is the farthest from the neutral axis.
- The displacement of the beam is always perpendicular to the neutral axis.

Equilibrium equations.



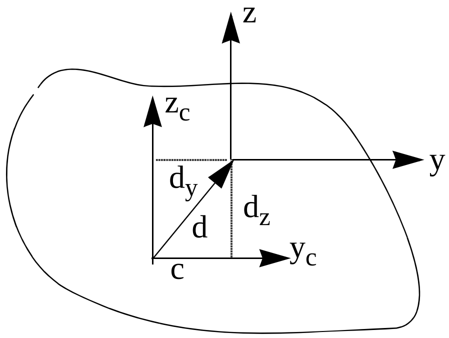


$$\frac{d\boldsymbol{V}_y}{dx} = -p_y \qquad \frac{d\boldsymbol{V}_z}{dx} = -p_z$$

$$\frac{d\boldsymbol{M}_z}{dx} = -V_y \qquad \frac{d\boldsymbol{M}_y}{dx} = -\boldsymbol{V}_y$$

Area Moment of Inertias

Parallel axis theorem



$$I_{yy} = I_{y_c y_c} + A d_z^2$$

$$I_{zz} = I_{z_c z_c} + A d_y^2$$

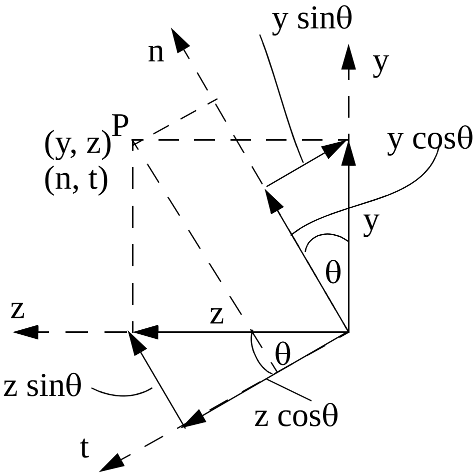
$$I_{yz} = I_{y_c z_c} + A d_y d_z$$

- I_{yy} and I_{zz} are always positive and minimum about the axis passing through the centroid of the body.
- I_{yz} can be positive or negative.
- If either y or z axis is an axis of symmetry then I_{yz} will be zero.

Coordinate Transformation

Definition 1 The coordinate system in which the cross moment of inertia is zero is called the principal coordinate system.

Definition 2 The moment of inertias in the principal coordinate system are called principal moment of inertias.



$$n = y\cos\theta + z\sin\theta$$
$$t = -y\sin\theta + z\cos\theta$$

$$I_{nn} = \int_A t^2 dA = I_{yy}\cos^2\theta + I_{zz}\sin^2\theta - 2I_{yz}\cos\theta\sin\theta$$

$$I_{tt} = \int_A n^2 dA = I_{yy}\sin^2\theta + I_{zz}\cos^2\theta + 2I_{yz}\cos\theta\sin\theta$$

$$I_{nt} = \int_A nt\,dA = (I_{yy} - I_{zz})\cos\theta\sin\theta + I_{yz}(\cos^2\theta - \sin^2\theta)$$

$$\tan 2\theta_p = \frac{-2I_{yz}}{(I_{yy} - I_{zz})} \qquad I_{1,2} = \frac{(I_{yy} + I_{zz})}{2} \pm \sqrt{\left(\frac{I_{yy} - I_{zz}}{2}\right)^2 + I_{yz}^2}$$

- Area moment of intertias are second order tensors.

6.5 The internal bending moments on the cross-section shown in Figure P6.5 were determined to be $M_y = -10 \; in-kips$ and $M_z = -12 \; in-kips$. Determine (a) the orientation of the neutral axis. (b) the maximum bending normal stress.

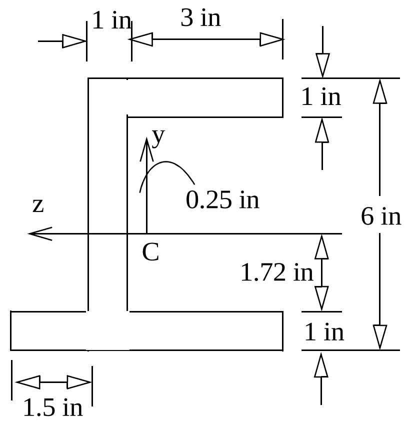


Figure P6.5

6.12 A cantilever beam is loaded such that there is no twist. The distributed load acts in the y-z plane at an angle of 24^0 from the x-y plane as shown in Figure P6.12. On a section at x = 60 in, determine: (a) the orientation of the neutral axis. (b) the bending normal stress at points A and B.

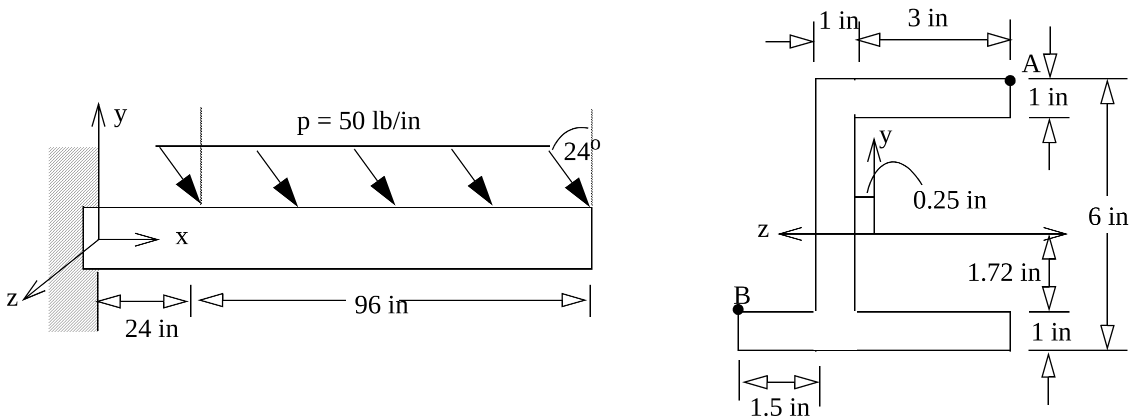
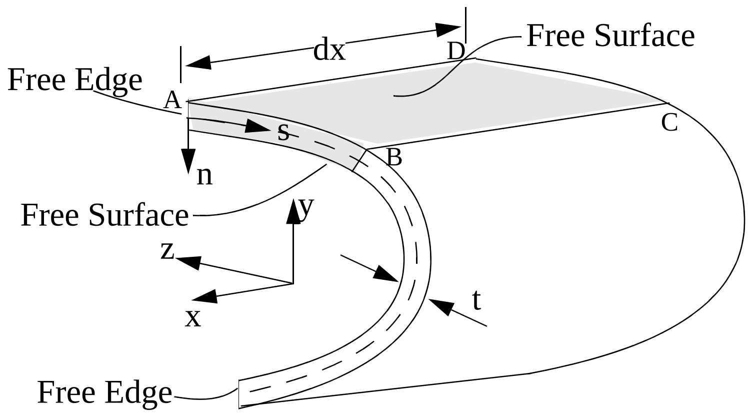


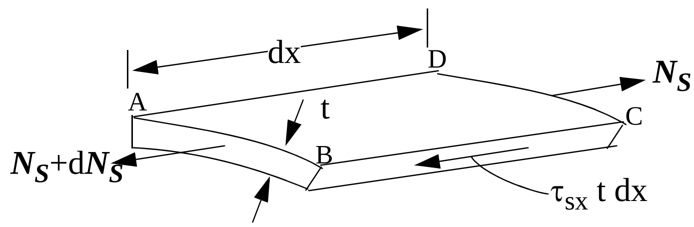
Figure P6.12

6.13 The modulus of elasticity for the beam in problem 6.12 is E = 30,000ksi. Determine the deflection of the beam at x = 60 inch and show that it is perpendicular to the neutral axis.

Shear stress in thin open sections

A differential element of a thin open section.





$$(N_S + dN_S) - N_S + \tau_{sx} t dx = 0 \text{ or}$$

Equlbrium Equations: $$\tau_{sx} t = -\frac{dN_s}{dx}$$

Axial Force: $$N_s = \int_{A_s} \sigma_{xx} \, dA$$

$$\tau_{sx} t = -\frac{d}{dx}\int_{A_s} \sigma_{xx} \, dA$$

Definition 3 The direction of the s-coordinate is from the free surface towards the point where shear stress is being calculated.

Definition 4 The area A_S is the area between free edge and the point at which the shear stress is being evaluated.

$$\tau_{sx} t = -\frac{d}{dx}\int_{A_s}\left[-\left(\frac{I_{yy}M_z - I_{yz}M_y}{I_{yy}I_{zz} - I_{yz}^2}\right)y - \left(\frac{I_{zz}M_y - I_{yz}M_z}{I_{yy}I_{zz} - I_{yz}^2}\right)z\right] dA$$

$$\tau_{sx} t = \frac{d}{dx}\left[\left(\frac{I_{yy}M_z - I_{yz}M_y}{I_{yy}I_{zz} - I_{yz}^2}\right)\int_{A_s} y dA + \left(\frac{I_{zz}M_y - I_{yz}M_z}{I_{yy}I_{zz} - I_{yz}^2}\right)\int_{A_s} z dA\right]$$

We define the first moment of the area A_S as:

$$Q_z = \int_{A_s} y\,dA \qquad Q_y = \int_{A_s} z\,dA$$

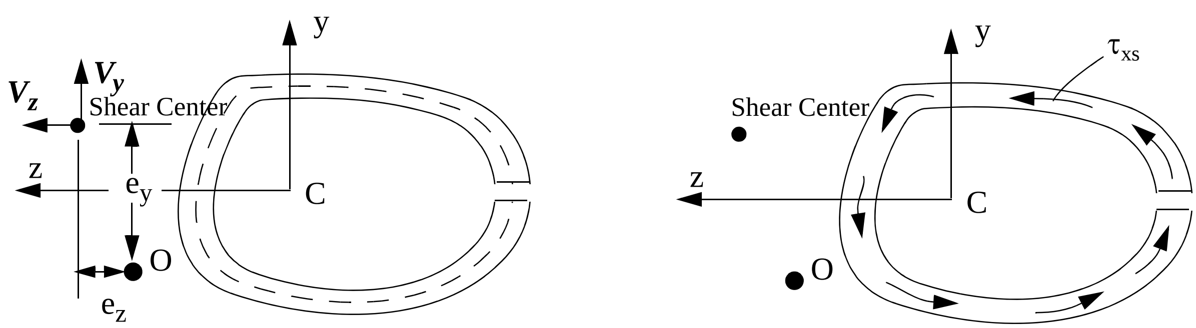
Assumption 11 The beam is not tapered.

$$\tau_{sx}t = \left(\frac{I_{yy}\frac{dM_z}{dx} - I_{yz}\frac{dM_y}{dx}}{I_{yy}I_{zz} - I_{yz}^2}\right)Q_z + \left(\frac{I_{zz}\frac{dM_y}{dx} - I_{yz}\frac{dM_z}{dx}}{I_{yy}I_{zz} - I_{yz}^2}\right)Q_y$$

$$\boxed{q = \tau_{sx}t = -\left(\frac{I_{yy}Q_z - I_{yz}Q_y}{I_{yy}I_{zz} - I_{yz}^2}\right)V_y - \left(\frac{I_{zz}Q_y - I_{yz}Q_z}{I_{yy}I_{zz} - I_{yz}^2}\right)V_z}$$

Shear center

From statics we know that any distributed force can be replaced by a force and a moment at any point, or, by a *single force (and no moment) at a specific point*. The specific point at which the shear stress (shear flow) can be represented by just shear forces V_y and V_z (components of a single force) and no internal torque is called the shear center. .



Definition 5 Shear center is a point in space at which the shear stress due to bending can be replaced by statically equivalent internal shear forces and no internal torque.

or

Shear center is a point in space such that if the line of action of external forces pass through the point then the cross-section will not twist.

6.24 A thin cross-section of uniform thickness t is shown in Figure P6.24. If shear stresses were to be found at points A and B what values of Q_y and Q_z are needed for the calculation. Assume $t \ll a$ and gap at D is of negligible thickness. Report the values of Q_y and Q_z in terms of t and a.

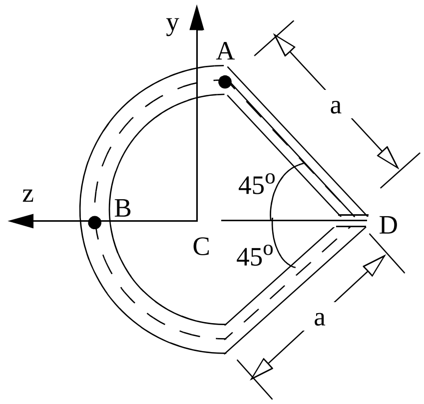


Figure P6.24

6.25 Shear forces on the cross-section shown in Figure P6.25 were calculated as $V_y = 10\ kips$ and $V_z = -5\ kips$. The cross section has a uniform thickness of 1/8 in. Determine the bending shear stresses at points A and B and report your answers as τ_{xy} and τ_{xz}.

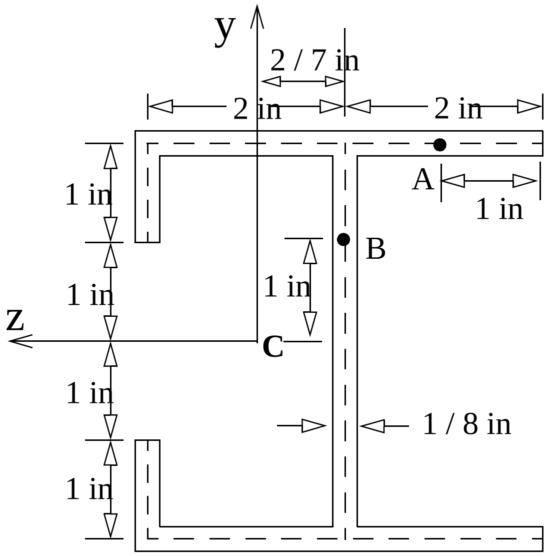


Figure P6.25

6.27 A cantilever beam is loaded such that there is no twist. The cross-section has a uniform thickness of 0.5 inch. Calculations show that $I_{yy} = 80.25\ in^4$ and $I_{zz} = 166.00\ in^4$. Determine the bending normal stress and bending shear stress at point A.

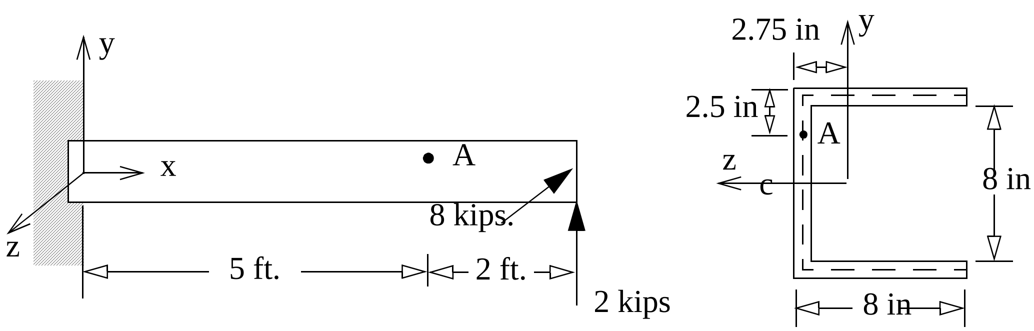


Figure P6.27

6.33 The cross-sections shown in Figure P6.33 has a uniform thickness t. Assuming $t \ll a$ determine the location of shear center with respect to point A

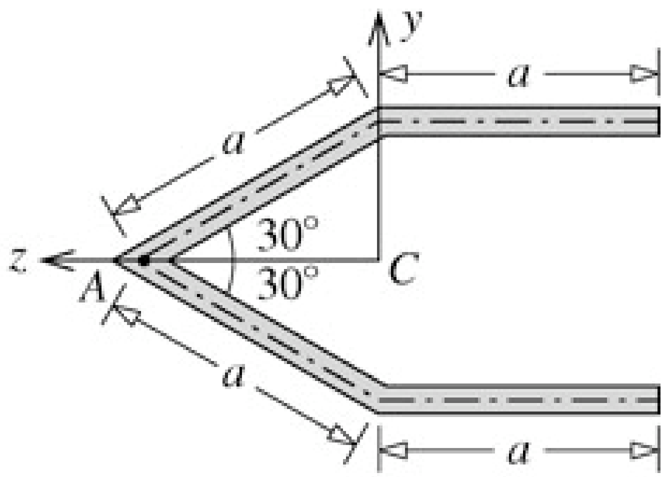


Figure P6.33

6.36 The cross-section shown in Figure P6.36 has a uniform thickness t. Assuming $t \ll a$ determine the location of shear centers with respect to point A.

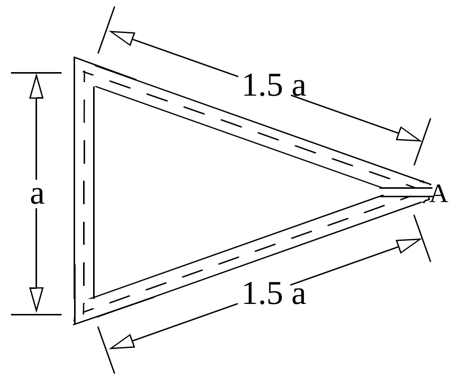
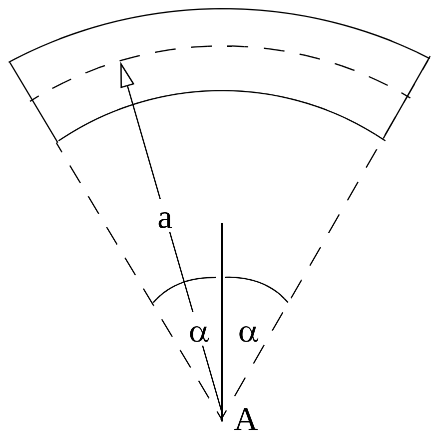


Figure P6.36

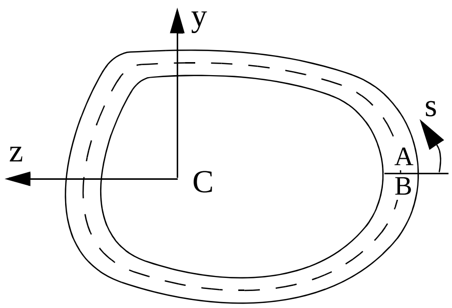
6.38 The cross-section shown in Figure P6.38 has a uniform thickness t and boundaries made from circular arcs. Assuming $t \ll a$ determine the location of shear centers with respect to point A in terms of radius a and angle α.

Figure P6.38

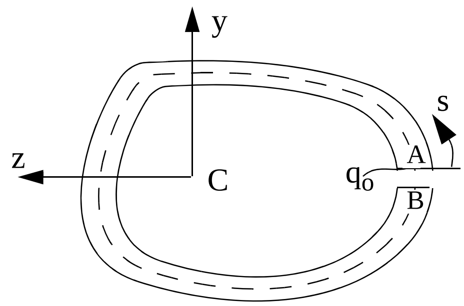


Shear stresses in thin closed sections

(a) **Thin closed section.**



(b) **An imaginary cut in closed section.**



$$q_c = q_o + q$$

q_c is the shear flow in the closed section at any point,

q is the shear flow of the open section, and

q_o is the unknown shear flow at the starting point that has to be determined.

Shear strain can be written as:

$$\gamma_{xs} = \frac{\partial u}{\partial s} + \frac{\partial v_s}{\partial x} = \frac{\tau_{xs}}{G}$$

u and v_s are displacement in the x and s direction, respectively, and

G is the shear modulus of elasticity.

$$\int_{s_A}^{s_B} \frac{\partial u}{\partial s} ds = \oint\left[\frac{\tau_{xs}}{G} - \frac{\partial v_s}{\partial x}\right] ds \quad \text{or} \quad u(s_B) - u(s_A) = \oint\left[\frac{\tau_{xs}}{G} - \frac{\partial v_s}{\partial x}\right] ds$$

Assumption 1 through Assumption 3 implies: Cross-section shape and dimension undergoes negligible change. This implies that no point on the cross-section moves relative to the other in the s-direction i.e., $v_s = 0$ in pure bending.

Noting that $u(s_B) = u(s_A)$ we obtain:

$$\oint\left(\frac{q_c}{t}\right) ds = \oint\left(\frac{q_o + q}{t}\right) ds = 0$$

If the thickness is uniform across the cross-section.

$$q_o = -\frac{1}{S}\oint q\, ds$$

where, S is the total path length of the perimeter of the cross-section.

6.47 The thin cross-section shown in Figure P6.47 is subjected to a shear force $V_y = V$ acting through the shear center. Determine the shear stress at points A and B in terms of V, a, and t.

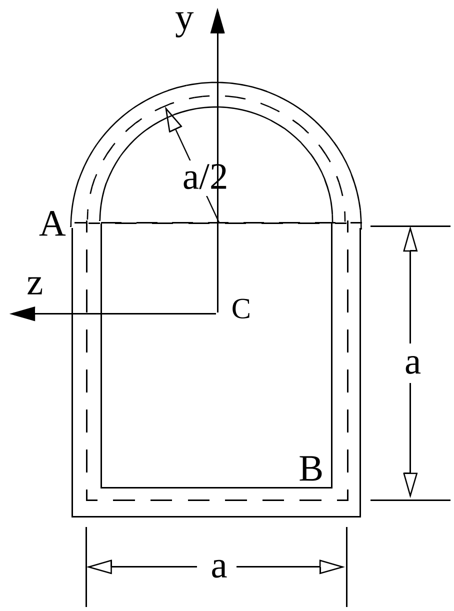


Figure P6.47

6.48 The thin cross-section shown is subjected to a shear force $V_z = V$ acting through the shear center. Starting with point A determine the shear stress at points A and B in terms of V, a, and t.

6.49 Determine the shear center of the cross-section shown in Figure P6.47.

6.67 A cantilever beam is loaded as shown in Figure P6.67. The cross-section has a uniform thickness of t = 1/4 in. Determine the normal and shear stress at points A and B in cartesian coordinates on a section next to the wall.

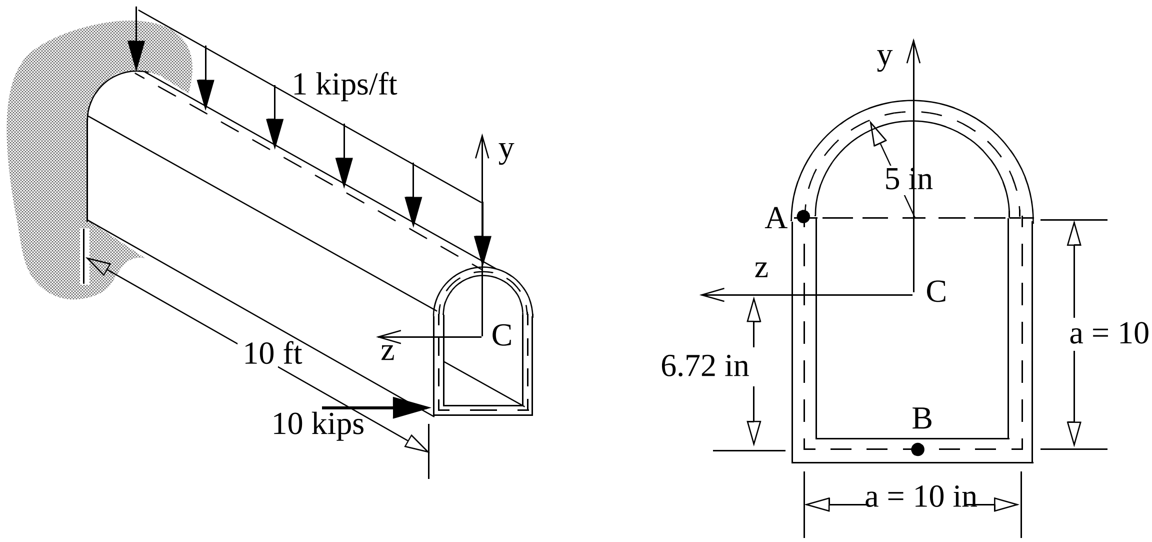


Figure P6.67