Transformation Matrices

Moving, Rotating, Stretching and Reflecting Parts.

What are they? What are they good for?

- In CAD, as well as in dynamics, kinematics, solid mechanics, and many other fields, it is necessary to know the **positions** and **orientations** of things.
- In CAD, we need to know the position and orientation of:
  - Primitive solids & features in parts
  - Work coordinate system
  - Components in assemblies
  - Visualization
- There are several ways of representing positions and orientations. In CAD, we mostly use "**transformation matrices**."

Defining a point as a vector

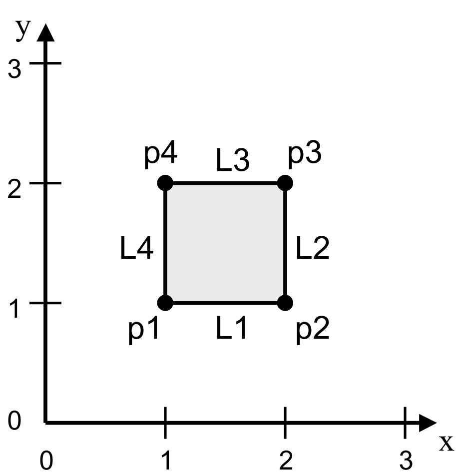
As column vectors

In 2D: $p = \begin{bmatrix} x \\ y \end{bmatrix}$ In 3D: $p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Using homogeneous coordinates

In 2D: $p = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ In 3D: $p = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

Representing a square

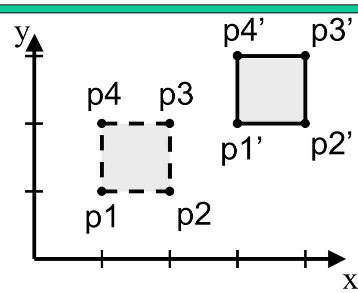


L1 = < p1, p2 >
L2 = < p2, p3 >
L3 = < p3, p4 >
L4 = < p4, p1 >

p1 = $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ p2 = $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

p3 = $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ p4 = $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Translating the square



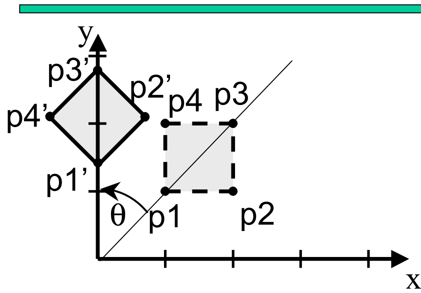
$$p1' = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad p2' = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$p3' = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \quad p4' = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$$p' = \mathbf{M}p = \mathit{Trans}(\ x,\ \ y) \bullet p = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p1' = \mathit{Trans}(2,1) \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Rotating about the origin



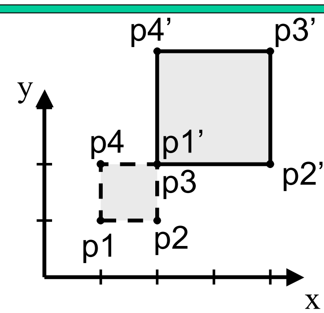
$$p1' = \begin{bmatrix} 0 \\ 1.414 \\ 1 \end{bmatrix} \quad p2' = \begin{bmatrix} .707 \\ 2.121 \\ 1 \end{bmatrix}$$

$$p3' = \begin{bmatrix} 0 \\ 2.828 \\ 1 \end{bmatrix} \quad p4' = \begin{bmatrix} -.707 \\ 2.121 \\ 1 \end{bmatrix}$$

$$p' = \mathbf{M}p = \mathit{Rot}(\theta) \bullet p = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p1' = \mathit{Rot}(45^o) \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} .707 & -\ .707 & 0 \\ .707 & .707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.414 \\ 1 \end{bmatrix}$$

Scaling



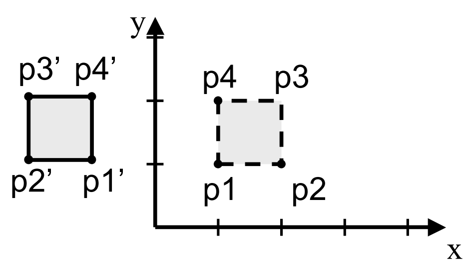
$$p1' = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad p2' = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$p3' = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} \quad p4' = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$p' = \mathbf{M}p = \mathit{Scale}(sx,sy) \bullet p = \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p1' = \mathit{Scale}(2, 2) \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Mirror Reflection



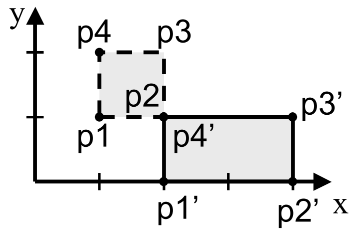
$$p1' = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad p2' = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$p3' = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \quad p4' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$p' = \mathbf{M}p = \mathit{MirrorX} \bullet p = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p1' = \mathit{MirrorX} \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Concatenation



$$p1' = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad p2' = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

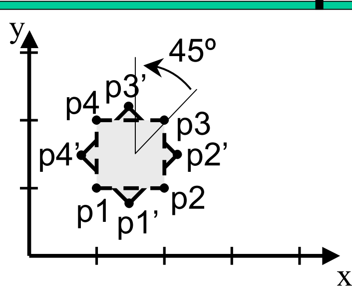
$$p3' = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \qquad p4' = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$p' = \mathbf{M_2 M_1} p = \mathit{Trans}(0,-1) \bullet \mathit{Scale}(2,1) \bullet p$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p1' = \mathit{Trans}(0,-1) \bullet \mathit{Scale}(2,1) \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Rotating about an arbitrary point



1.

2.

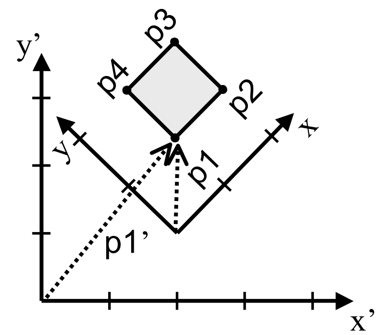
3.

$p' = \mathbf{M_3 M_2 M_1} p$

$= \qquad\qquad\qquad\qquad\qquad\qquad p$

$$= \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix} \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix} \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Performing a "Mapping"



$$p1' = \begin{bmatrix} ? \\ ? \\ 1 \end{bmatrix} \quad p2' = \begin{bmatrix} ? \\ ? \\ 1 \end{bmatrix}$$

$$p3' = \begin{bmatrix} ? \\ ? \\ 1 \end{bmatrix} \quad p4' = \begin{bmatrix} ? \\ ? \\ 1 \end{bmatrix}$$

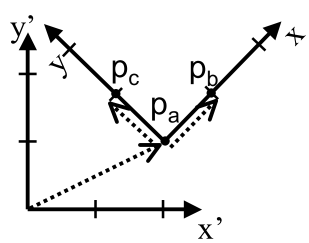
p1', p2', p3' & p4' are in the same spot as p1, p2, p3 & p4, respectively, but have coordinates with respect to the x' and y' coordinate system.

The mapping equation is: $p' = \mathbf{T}p = \begin{bmatrix} n_x & o_x & p_x \\ n_y & o_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

How do we solve for the Mapping Matrix?

Substitute $p_a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad p_a' = \begin{bmatrix} n_x & o_x & p_x \\ n_y & o_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$

Substitute $p_b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad p_b' = \begin{bmatrix} n_x & o_x & p_x \\ n_y & o_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} n_x + p_x \\ n_y + p_y \\ 1 \end{bmatrix}$



$$p_b' - p_a' = \begin{bmatrix} n_x + p_x \\ n_y + p_y \\ 1 \end{bmatrix} - \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix}$$

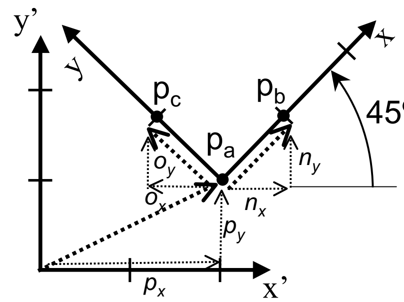
Similarly $p_c' - p_a' = \begin{bmatrix} o_x \\ o_y \\ 0 \end{bmatrix}$

Solving for the Mapping Matrix in our example

$$\begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = p_a' = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

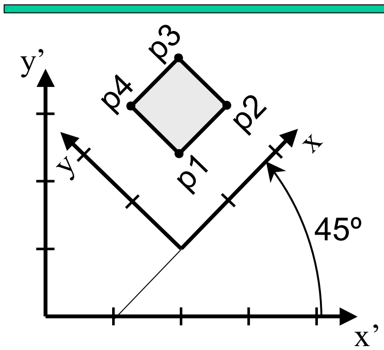
$$\begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix} = p_b' - p_a' = \begin{bmatrix} .707 \\ .707 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} o_x \\ o_y \\ 0 \end{bmatrix} = p_c' - p_a' = \begin{bmatrix} -.707 \\ .707 \\ 0 \end{bmatrix}$$



$$\mathbf{M} = \begin{bmatrix} n_x & o_x & p_x \\ n_y & o_y & p_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing the "Mapping"



$$p' = \mathbf{T}p = \begin{bmatrix} .707 & -.707 & 2 \\ .707 & .707 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p1' = \begin{bmatrix} 2 \\ 2.414 \\ 1 \end{bmatrix} \quad p2' = \begin{bmatrix} 2.707 \\ 3.121 \\ 1 \end{bmatrix}$$

$$p3' = \begin{bmatrix} 2 \\ 3.828 \\ 1 \end{bmatrix} \quad p4' = \begin{bmatrix} 1.293 \\ 3.121 \\ 1 \end{bmatrix}$$

3D Transformation Matrices

Translation

$$p' = \mathbf{M}p = \mathit{Trans}(\ x,\ \ y,\ \ z) \bullet p = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Scaling

$$p' = \mathbf{M}p = \mathit{Scale}(s_x,s_y,s_z) \bullet p = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Transformation Matrices

Mirror in X direction (about YZ plane)

$$p' = \mathbf{M}p = \mathit{MirrorX} \bullet p = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Mirror in Y direction (about ZX plane)

$$p' = \mathbf{M}p = \mathit{MirrorY} \bullet p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Mirror in Z direction (about XY plane)

$$p' = \mathbf{M}p = \mathit{MirrorZ} \bullet p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Transformation Matrices

Rotate about Z axis

$$p' = \mathbf{M}p = RotZ(\theta) \bullet p = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotate about X axis

$$p' = \mathbf{M}p = RotX(\theta) \bullet p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotate about Y axis

$$p' = \mathbf{M}p = RotY(\theta) \bullet p = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

What you should know:

- Transformation matrices for translation, rotation, scaling and mirroring in 2D
- Concatenation
- Mappings
  - what they are used for