FEA Solution Procedure

(demonstrated with a 1-D bar
element problem)

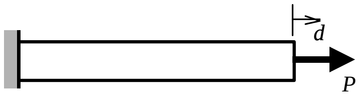
FEA Procedure for Static Analysis

1. Prepare the FE model
    a. discretize (mesh) the structure
    b. prescribe loads
    c. prescribe supports
2. Perform calculations (solve)
    a. generate stiffness matrix ($\mathbf{k}$) for each element
    b. connect elements (assemble $\mathbf{K}$)
    c. assemble loads (into load vector $\mathbf{R}$)
    d. impose supports conditions
    e. solve equations ($\mathbf{KD}=\mathbf{R}$) for displacements
3. Postprocess

Deflection of a Bar Element

- If we fix the left end of a bar (with constant cross section) it's end deflection is given by:

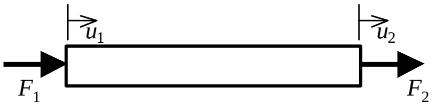
$$d = \frac{PL}{AE}$$



- If the left end is NOT fixed, the relationship between force and deflection is given by:

$$\frac{AE}{L}\left(u_1 - u_2\right) = F_1$$

$$\frac{AE}{L}\left(-u_1 + u_2\right) = F_2$$



Deflection of a Bar Element

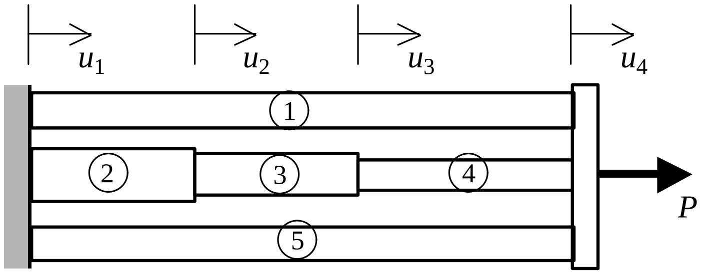
- These two equations can be conveniently expressed in matrix form as:

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

- The different parts are known as:
  - the elemental stiffness matrix $\mathbf{k} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
  - the elemental displacement vector $\mathbf{d} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$
  - the elemental force vector $\mathbf{r} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$
- This form allows us to easily combine the equations from all elements of a structure.

The 1-D Bar Problem

E.g.



Calculate:
- deflections,
- strains,
- stresses, and
- internal forces.

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a. Generate Elemental Equations

$$\frac{A_1E_1}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_{1,1}\\ F_{4,1}\end{Bmatrix}\qquad \frac{A_2E_2}{L_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}F_{1,2}\\ F_{2,2}\end{Bmatrix}$$

Node number ⟶ ⟵ Element number

$$\frac{A_3E_3}{L_3}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}F_{2,3}\\ F_{3,3}\end{Bmatrix}\qquad \frac{A_4E_4}{L_4}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_{3,4}\\ F_{4,4}\end{Bmatrix}$$

$$\frac{A_5E_5}{L_5}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_{1,5}\\ F_{4,5}\end{Bmatrix}$$

b. & c. Combine Equations

The global set of equations is obtained by summing the equations for each force.

In matrix form, the result is:

$$\begin{bmatrix} \frac{A_1E_1}{L_1}+\frac{A_2E_2}{L_2}+\frac{A_5E_5}{L_5} & -\frac{A_2E_2}{L_2} & 0 & -\frac{A_1E_1}{L_1}-\frac{A_5E_5}{L_5} \\ -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2}+\frac{A_3E_3}{L_3} & -\frac{A_3E_3}{L_3} & 0 \\ 0 & -\frac{A_3E_3}{L_3} & \frac{A_3E_3}{L_3}+\frac{A_4E_4}{L_4} & -\frac{A_4E_4}{L_4} \\ -\frac{A_1E_1}{L_1}-\frac{A_5E_5}{L_5} & 0 & -\frac{A_4E_4}{L_4} & \frac{A_1E_1}{L_1}+\frac{A_4E_4}{L_4}+\frac{A_5E_5}{L_5} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ P \end{Bmatrix}$$

Combined Equations

The different parts are known as:

- the global stiffness matrix $(\mathbf{K})$
- the global displacement vector $(\mathbf{D})$
- the global force vector $(\mathbf{R})$

I.e.: $\mathbf{KD=R}$

Combined Loads

- Note also that the internal forces are balanced by the external applied loads and reactions:

$$\begin{Bmatrix} F_{1,1}+F_{1,2}+F_{1,5} \\ F_{2,2}+F_{2,3} \\ F_{3,3}+F_{3,4} \\ F_{4,1}+F_{4,4}+F_{4,5} \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ P \end{Bmatrix}$$

- If there are no external loads on a node, the internal forces must balance to 0.

d. Support Conditions

- In this example we have one support condition: $u_1 = 0$.
- If we set $u_1 = 0$ in the global set of equations, then the first column of the stiffness matrix is not necessary, and the bottom 3 equations can be written as:

$$\begin{bmatrix} \frac{A_2E_2}{L_2}+\frac{A_3E_3}{L_3} & -\frac{A_3E_3}{L_3} & 0 \\ -\frac{A_3E_3}{L_3} & \frac{A_3E_3}{L_3}+\frac{A_4E_4}{L_4} & -\frac{A_4E_4}{L_4} \\ 0 & -\frac{A_4E_4}{L_4} & \frac{A_1E_1}{L_1}+\frac{A_4E_4}{L_4}+\frac{A_5E_5}{L_5} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix}$$

Support Conditions

- As well, the reaction at the boundary can be solved using the top row equation:

$$\left\langle -\frac{A_2E_2}{L_2} \quad 0 \quad -\frac{A_1E_1}{L_1} - \frac{A_5E_5}{L_5} \right\rangle \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = R_1$$

- But this must be done after u_2, u_3 and u_4 have been calculated.

e. Solving for Deflections

- The global matrices (with boundary condition rows/columns removed) are solved for deflections (**U**).

$$\mathbf{U}=\mathbf{K}^{-1}\mathbf{F}$$

- In practice, the computer does not actually calculate $\mathbf{K}^{-1}$, but solves for **U** directly, using some technique such as Guassian Elimination.

3.a. Solving for Strain

- Once we know the deflections, we can calculate the strain for each element.
- In a one dimensional problem, the strain is given by:

$$e = \frac{du}{dx}$$

- For the bar,

$$e = \frac{u_2 - u_1}{L}$$

3.b. Stress and Internal Force

- Below the yield stress, for a one-dimensional problem, stress is given by:

$$s = Ee$$

- The force in each bar can be calculated by:

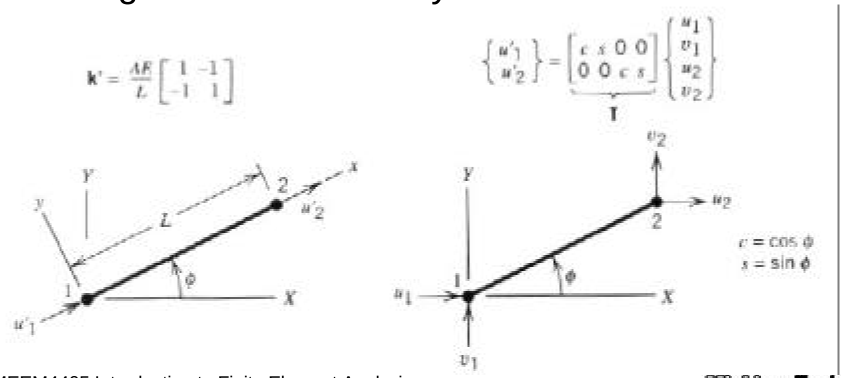
$$F = sA$$

or by:

$$F = \frac{AE(u_2 - u_1)}{L}$$

What about Planar (2-D) problems?

- The equation for a bar element with an arbitrary orientation in planar space is obtained by transforming the local element coordinate system to the global coordinate system.



$$\mathbf{k}' = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}}_{\mathbf{T}} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

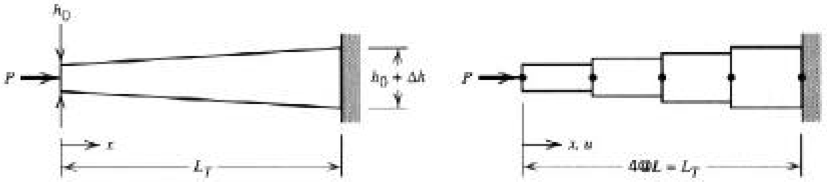
What about Planar (2-D) problems?

- Mathematically this is done by multiplying the elemental stiffness equation by a rotation matrix:

$$\mathbf{k} = \mathbf{T}^T\mathbf{k}'\mathbf{T} = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

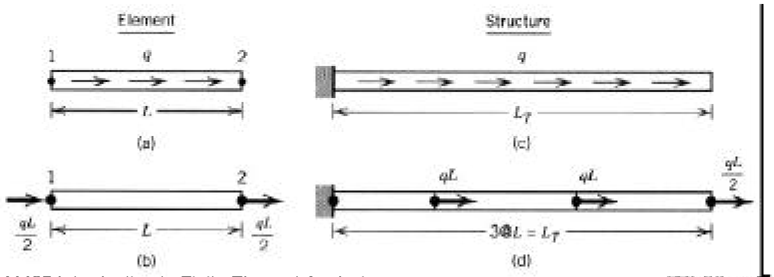
Varying cross-section

- A bar element with varying cross-section does not have constant strain, therefore $\delta = PL/AE$ can not be used.
- We could develop a new equation for a bar with non-uniform cross-section, but instead, we approximate the solution with a set of constant cross-section bar elements.



Distributed Load

- A bar element with distributed loading does not have constant strain. ($\delta = PL/AE$ can not be used.)
- We could develop a new equation for a bar with distributed loading, but in the Finite Element Method, we approximate the solution with a set of bar elements with loaded nodes in between.



Formal Procedure for Bar and Beam Elements

Another way to calculate the
stiffness matrix!

Bar Element – Formal Method

- We have shown how to obtain the elemental stiffness equations for a bar element using the direct method. We can also obtain these equations through a more general, formal procedure.

Bar Element – Formal Method

For most elements a general formula is used to calculate **k**,

$$\mathbf{k} = \int \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV$$

where **B** is the "strain-displacement matrix" and **E** is the "material property matrix."

To obtain **B** for a bar, we must first find $u(x)$.

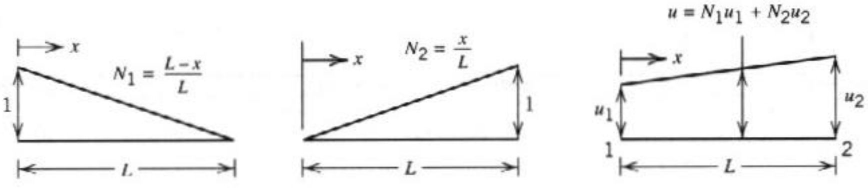
Bar Element – Formal Method

- Let us assume that

$$u = \begin{bmatrix} \dfrac{L-x}{L} & \dfrac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \text{or} \quad u = \mathbf{N}\mathbf{d}$$

(This is correct for a bar with constant cross-section and no distributed loads.)

- **N** is called the shape function matrix.



Bar Element – Formal Method

- The axial strain is given by:

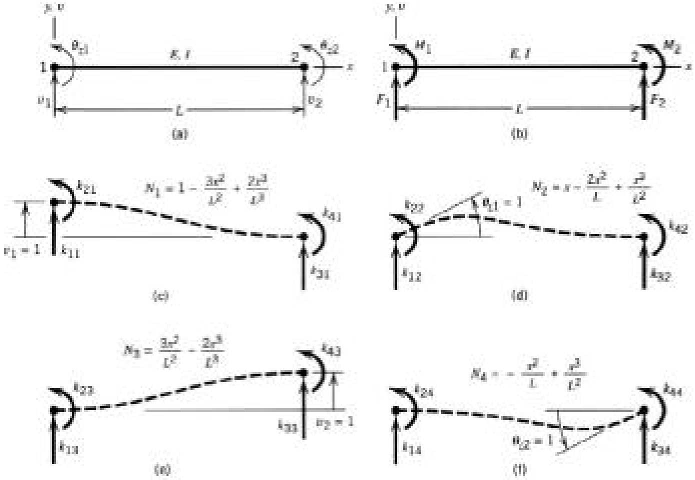
$$\varepsilon_x = \frac{du}{dx} = \left[\frac{d}{dx}\mathbf{N}\right]\mathbf{d} = \mathbf{Bd} \quad \text{where} \quad \mathbf{B} = \left[-\frac{1}{L} \quad \frac{1}{L}\right]$$

- Thus $e_x = (u_2 - u_1)/L$.
- $\boldsymbol{E}$ is simply the elastic modulus E (a scalar)
- dV is $A\,dx$, thus

$$\mathbf{k} = \int_0^L \begin{Bmatrix} -1/L \\ 1/L \end{Bmatrix} E \left[-\frac{1}{L} \quad \frac{1}{L}\right] A\,dx = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Beam Element – Direct Method

- We start with the shape functions.



Beam Element – Direct Method

- There are two degrees of freedom (displacements) at each node: v and θ_z. Each shape function corresponds to one displacement equal to one and all the others equal to zero.
- Note that everything we do in this course assumes that the displacements are small.

Beam Element – Direct Method

- Using standard beam deflection formulae and statics, we solve for one column of $\mathbf{k}$ at a time.
- E.g., to solve for column 1 of $\mathbf{k}$:

$v_1 = 1$ at node 1 $\qquad 1 = \dfrac{k_{11}L^3}{3EI} - \dfrac{k_{21}L^2}{2EI}$

$\theta_{z1} = 0$ at node 1 $\qquad 0 = -\dfrac{k_{11}L^2}{2EI} + \dfrac{k_{21}L}{EI}$

$\sum(\text{forces})_y = 0 \qquad 0 = k_{11} + k_{31}$

$\sum(\text{moments})_{\text{node 2}} = 0 \qquad 0 = k_{21} + k_{41} - k_{11}L$

Beam Element – Direct Method

The result is:

$$\mathbf{k} = \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix}$$

which operates on $\mathbf{d} = [v_1, \theta_{z1}, v_2, \theta_{z2}]^T$.

Beam Element – Formal Method

- The formal beam element stiffness matrix derivation is much the same as the bar element stiffness matrix derivation. We start with the formula:

$$\mathbf{k} = \int_0^L \mathbf{B}^T EI\, \mathbf{B}\, dx$$

- The commonality is that $\mathbf{d}^T\mathbf{k}\mathbf{d}/2$ gives the strain energy.

Beam Element – Formal Method

$$v = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3$$

$$v = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix} = \mathbf{N}\mathbf{d}$$

$$\frac{d^2 v}{dx^2} = \left[\frac{d^2}{dx^2} \mathbf{N} \right] \mathbf{d} = \mathbf{B}\mathbf{d}$$

Beam Element – Formal Method

$$\mathbf{B} = \left[-\frac{6}{L^2} + \frac{12x}{L^3} \quad -\frac{4}{L} + \frac{6x}{L^2} \quad \frac{6}{L^2} - \frac{12x}{L^3} \quad -\frac{2}{L} + \frac{6x}{L^2} \right]$$

- Stress is given by:

$$s_x = My / I$$

$$M = EI \frac{d^2 v}{dx^2} = EI\, \mathbf{B}\mathbf{d}$$

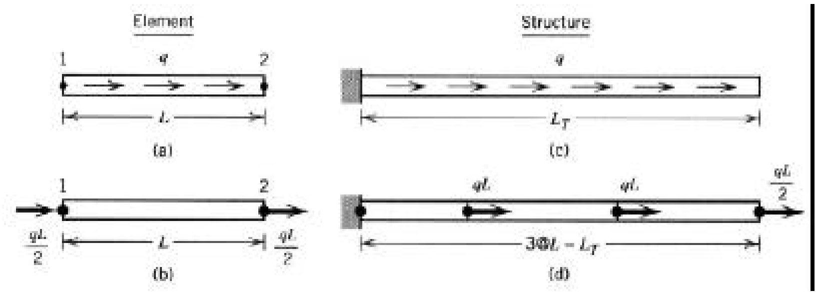
Beam Element w/Axial Stiffness

- If we combine the above bar and beam stiffness matrices, we get a general beam stiffness matrix with axial stiffness.

$$\mathbf{k} = \left[\begin{array}{cccccc} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{array}\right] \begin{array}{c} u_1 \\ \upsilon_1 \\ \theta_{z1} \\ u_2 \\ \upsilon_2 \\ \theta_{z2} \end{array}$$

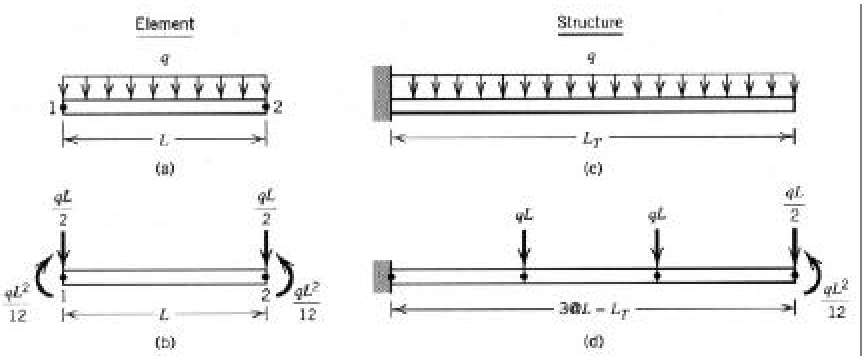
Uniformly Distributed Loads

can be represented by equivalent loads at intermediate nodes. Axially:



Uniformly Distributed Loads

Laterally:



Orientating Element in 3-D Space

- Transformation matrices are used to transform the equations in the element coordinate system to the global coordinate system, as was shown for the bar element in 2-D planar space.