Application of maximum likelihood methods to laser Thomson scattering measurements of low density plasmas

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Laser Thomson scattering (LTS) is an established plasma diagnostic technique that has seen recent application to low density plasmas. It is difficult to perform LTS measurements when the scattered signal is weak as a result of low electron number density, poor optical access to the plasma, or both. Photon counting methods are often implemented in order to perform measurements in these low signal conditions. However, photon counting measurements performed with photo-multiplier tubes are time consuming and multi-photon arrivals are incorrectly recorded. In order to overcome these shortcomings a new data analysis method based on maximum likelihood estimation was developed. The key feature of this new data processing method is the inclusion of non-arrival events in determining the scattered Thomson signal. Maximum likelihood estimation and its application to Thomson scattering at low signal levels is presented and application of the new processing method to LTS measurements performed in the plume of a 2-kW Hall-effect thruster is discussed. © 2013 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4821980]

I. INTRODUCTION

Lasers have been used to measure plasma properties since shortly after they were developed in practical form in the 1960s.¹ One of the most popular ways to measure electron properties in a plasma is laser Thomson scattering (LTS). Thomson scattering is the process by which photons are elastically scattered from the free electrons of a plasma.² Since the electrons have thermal energy their motion causes a Doppler shift in the scattered photons that is proportional to their velocity along the scattering vector, the direction of which is determined by the incident laser beam and the location of the collection optics. Because of this LTS actually measures the electron velocity distribution function directly. The number of scattered photons detected can be calibrated to determine the density of free electrons in the plasma.

The proliferation of ruby lasers made Thomson scattering a widespread diagnostic tool in plasma physics,³ most notably in the fusion community. Fusion plasmas have high electron density and temperature which produce a strong Thomson scattering signal. High temperatures produce a broad scattered signal easily distinguishable from the laser light and high density produces a strong signal that can easily be seen over background noise sources. LTS measurements on the T-3 Tokamak in the U.S.S.R.⁴ produced measurements of electron temperature ranging from 100 eV to 1000 eV with densities of (1–3) × 10¹⁹ m⁻³.

In the mid 1980s Thomson scattering was extended to colder processing plasmas (T_e = 1–10 eV) with moderate electron densities (10¹⁸–10¹⁹ m⁻³). Performing Thomson scattering measurements in this domain has a different set of challenges compared to fusion plasmas. Plasmas with high temperature and density have large amounts of line emission, Bremsstrahlung, and stray light present as noise sources. For cold plasmas Bremsstrahlung is typically negligible, and there is a lack of strong line emission at most relevant wavelengths. As a logical extension of fusion work, early measurements on cold plasmas continued to use ruby lasers. Examples of this are measurements on an inductively coupled plasma (ICP),⁵, ⁶ measurements on a low-pressure plasma spray (LPPS) device,⁷ and the first LTS measurements of an electron cyclotron resonance (ECR) plasma.⁸

Contemporary with early work on processing plasmas, Nd:YAG lasers with high pulse energy and high repetition rates were being developed along with visible-wavelength detectors with high sensitivity and low noise.⁹ These lasers combine good power and pointing stability, low beam divergence, and good beam quality,¹⁰ and as a result frequency doubled Nd:YAG lasers operating at 532 nm have become the dominant laser source for Thomson scattering measurements. Electron temperature and density in many devices have since been measured using Nd:YAG lasers, including expanding arc plasmas,¹¹, ¹² ECR discharges,¹³–¹⁵ and microwave plasma torches.¹⁶, ¹⁷

Until the late 1990s most LTS diagnostics on cold plasmas used photomultiplier tubes (PMTs) to detect the scattered Thomson signal. PMTs have excellent sensitivity and the ability to count single photons, but they also have the disadvantage of being able to detect only one wavelength of scattered light at a time, necessitating measurements at multiple wavelengths to construct a full scattered spectrum (or use of PMT arrays to detect the full spectrum). Use of the wavelength scanning technique with a single PMT is time consuming, especially when low electron density necessitates acquisition times of 30 min or more at each wavelength. The desire for shorter acquisition times led to the use of intensified charge-couple devices (iCCDs) to collect the entire scattered spectrum at once, greatly reducing acquisition times. In one experiment an iCCD was combined with a multi-pass cell and low-power/high rep-rate Nd:YAG laser, resulting in a detection limit of approximately 10¹⁷ m⁻³ for acquisition times of about 25 min.¹⁸
Beginning in the late 2000s Thomson scattering measurements were performed inside the discharge chamber of a miniature microwave ion thruster. A triple grating spectrograph and PMT were used for this work, and the data were processed using a photon counting method. Electron densities of approximately $10^{18}$ m$^{-3}$ were measured, and electron temperature ranged from about 2 to 10 eV. These measurements were the first application of LTS to a space propulsion device. Other space propulsion devices (such as arcjets, Hall thrusters, and ion thrusters) have electron temperatures and densities in a similar range ($1 \times 10^{16} - 5 \times 10^{18}$ m$^{-3}$, 1–50 eV). These compact electric space propulsion devices are used on a variety of space missions including station keeping, orbit transfer, and primary propulsion for deep space missions.

LTS measurements of the electron properties in the plumes of space propulsion devices can be difficult as a result of low electron density and difficult optical access to the plasma. Due to the combination of these two factors the number of scattered photons detected per laser pulse is so low that it is necessary to measure scattering from thousands of laser pulses in order to determine the scattered Thomson spectrum. It is typical to use photon counting methods in order to “build up” the scattered spectrum in this situation of extremely low photon flux, but such methods have two significant drawbacks. First, they require time-intensive wavelength scanning in order to determine the entire scattered spectrum (this drawback is removed when using PMT arrays or iCCDs with photon-counting modes). Second, only photon arrivals are considered statistically significant, which results in disregarding a large fraction of the acquired data. For this work a new method of data processing was developed to handle the exceptionally small photon fluxes that were detected while using an iCCD camera in order to measure the entire scattered spectrum at once. The data are processed using a maximum likelihood estimation algorithm, and non-arrival events are included in the data processing in order to improve the estimate of the mean photon arrival rate as a function of wavelength.

In this paper the apparatus developed to perform laser measurements of electron temperature and density in the plume of plasma thrusters will be detailed in Sec. II. Data acquisition and the new maximum likelihood estimation data processing method will be discussed in Secs. III and IV, respectively. Sample results from measurements performed on a 2-kW Hall thruster will be presented in Sec. V. Finally, a summary of key results will be presented in Sec. VI.

II. EXPERIMENTAL APPARATUS

A full schematic of the measurement apparatus can be seen in Figure 1. The Hall-effect thruster used for this work was a 2-kW-class thruster similar to an Aerojet BPT-2000. The outer diameter of the thruster body is 120 mm and the channel width is 13 mm. Nominal power input is 2200 W at 350 V, which yields a specific impulse of 1765 s (operating on xenon) with approximately 50% efficiency. The thruster weighs 5.3 kg and was operated at a variety of mass flow rates using xenon as the propellant gas.

A Quantel YG980 Q-switched Nd:YAG laser with second harmonic generation was used for this work. Output energy is 610 mJ at 532 nm with a pulse duration of 6 ns (full width at half-maximum) and a pulse repetition rate of 10 Hz. A 1-in.-diameter plano-convex lens focuses the beam into the plasma, with the focal point located on the discharge channel center-line 10 mm downstream from the thruster exit plane. The diameter of the beam at the focus is approximately 375 μm and the length of the scattering volume was 6 mm. After passing through the plume the beam was dissipated in a custom beam dump using black glass at Brewster’s angle.

Two lenses in infinite-conjugate configuration were used to collect and focus the scattered light onto the spectrograph.
entrance slit. The first lens collects and collimates the scattered light from the plasma. In order to maximize the signal passed to the detector the lens must collect as much light as possible. A 76.2-mm-diameter lens with a focal length of 100 mm was chosen for the collimating lens, which gives an extremely fast f/# of 1.3. The second lens has a focal length of 300 mm in order to match the spectrograph input solid angle. The spectrograph input slit was set to a height of 2 mm and width of 1 mm.

The spectrograph used for this work was a Spex® Triplemate 1877C-AG triple spectrograph. This imaging spectrograph is designed to provide a flat focal plane for use with two-dimensional imaging devices. The first stage consists of two Czerny-Turner monochromators with 600 g/mm gratings coupled in subtractive-dispersive mode. This pair acts as a tunable bandpass filter and passes a small spectral region of non-dispersed light to the final spectrograph stage. The final spectrograph has a 0.6 m length and is of asymmetrical Czerny-Turner design.

Data acquisition was performed with a Dicam Pro ICDD camera. The ICDD is cooled by a two-stage Peltier thermo-electric cooler to an operating temperature of −12°C in order to reduce the dark noise. The key element of an intensified CCD is the intensifier itself, which in addition to supplying gain to the signal can also be used to gate the detector. Gating the detector allows one to gather light only in a short window containing the laser pulse, greatly reducing the contribution of continuous background emission from the plasma. Control of the camera was performed using PCO Camware from the Cooke Corporation. This software suite allows detailed control over gating, triggering, binning, and data output. Due to the low density of the plasma investigated in this work and the poor quantum efficiency of the photocathode (only 10% at 532 nm), the 1024 × 1280 pixel detector was binned into 8 (H) × 32 (V) super-pixels. The horizontal binning was chosen such that signal was increased, but spectral resolution of approximately 0.1 nm was maintained. Vertical binning was chosen such that a super-pixel corresponds to a length of 0.5 mm along the scattering volume. The camera accepts a trigger input from the laser Q-switch via a BNC cable and then transmits the data from the camera to a computer via a fiber-optic cable.

III. DATA ACQUISITION

The entire scattering system is controlled with a dedicated computer system that controls both the camera and the laser. Since the camera is gated to only detect signal in a small window containing the laser pulse the camera requires an accurate trigger pulse. For a sequence of laser shots the trigger is provided by a synchronized output on the laser control unit. A BNC trigger line connected to the camera provides a time-advanced rising edge trigger to the camera, which then waits a fixed delay before opening the electro-optical shutter on the camera. The camera shutter is only open for 20 ns each pulse, allowing the capture of the 6 ns (FWHM) laser pulse. For measurements with the laser off, the camera is triggered by an Agilent 33120A digital function generator operating at the laser repetition rate of 10 Hz. The shutter times are the same as for laser measurements so that the plasma emission can be accurately subtracted from the total spectrum.

For all measurement conditions 27,000 acquisitions were taken, which was the maximum possible determined by the memory available in the camera. Each individual acquisition is stored in a separate ascii file containing a two-dimensional matrix the size of the detector face, with each unit in the matrix corresponding to a binned super-pixel. The value of each matrix unit is the number of counts recorded during that acquisition.

Thomson scattering spectra are always determined by taking the difference of at least two separate measurements. For this work two sets of laser data were taken at each thruster operating condition.

1. The first measurement was taken with the Hall-effect thruster operating at the nominal condition with the laser off. This data set contains the background emission spectrum of the thruster for that specific operating condition. Since the laser is off there is no stray light component. This will be referred to as the emission spectrum.

2. The second measurement was made with the thruster operating and the laser on. This measurement contains the full spectrum, including Rayleigh scattering, Thomson scattering, the stray light, and the background plasma emission. This will be referred to as the total spectrum.

As a result of this process there are two data sets made up of 27,000 files each, with each file containing a single acquisition. These files are used to create a single corrected spectrum using a procedure detailed in Sec. IV.

IV. DATA PROCESSING

A. Maximum likelihood estimation

Maximum likelihood estimation is a technique that can be used to estimate the parameters of a distribution from a sample of data using knowledge of the probability density function for the system under investigation. To begin, one must specify the distribution whose parameters are the object of estimation. Let \( x = \{x_1, x_2, \ldots, x_n\} \) represent a set of random measurements from an unknown population. The probability density function that characterizes this unknown distribution can be written generically as \( f(x|\mu) \), where the function \( f \) represents the probability of measuring the values in the vector \( x \), given the fact that the distribution is known to be characterized by the parameter \( \mu \). Since the observations \( x_i \) are randomly obtained, they can be considered to be statistically independent, since measuring \( x_1 \) in no way affects future measurements. By definition, for a statistically independent random variable the probability of obtaining the measurements \( x = \{x_1, x_2, \ldots, x_n\} \) can be expressed as the product of the individual probabilities of each measurement,\(^{27}\) given by

\[
f(x = \{x_1, x_2, \ldots, x_n\}|\mu) = f_1(x_1|\mu) \cdot f_2(x_2|\mu) \cdot \cdots f_n(x_n|\mu),
\]

(1)

where \( \mu \) can take on any value that satisfies the distribution \( f \). This statement is predictive in nature, in that it is assumed that if the form of \( f \) and its characteristic parameter \( \mu \) is known,
predictions can be made concerning measurements of this distribution. In reality, the problem is reversed in practical statistics.

In order to determine the solution to this inverse problem, the likelihood function can be defined by reversing the roles of the data and the distribution parameters, yielding:

$$L(\mu|x) = f(x|\mu) = \prod_{i=1}^{n} f(x_i|\mu).$$  \hspace{1cm} (2)

In this case the measurements, \( x \), are considered to be fixed and the value of \( \mu \) is varied while computing the value of \( L \). The value of \( \mu \) that maximizes this function is the maximum likelihood estimator of \( \mu \), often written as \( \hat{\mu} \).

Maximizing the likelihood function can be computationally difficult. For large samples calculating the likelihood for each estimate of the characteristic parameters can be time-consuming. Another problem that often occurs is that when computing the maximum likelihood as written in Eq. (2), one may end up with many very large or very small numbers that must be multiplied. Computers have difficulty handling very large and very small numbers, and the natural logarithm of the likelihood function is often maximized instead. In this case the convention is to refer to this new function as the log-likelihood, given by

$$\ln[L(\mu|x)].$$  \hspace{1cm} (3)

Since the logarithm is a monotone transform, the resulting value of the maximum likelihood estimator is the same whether the likelihood or log-likelihood function is maximized.

### B. MLE applied to LTS

Maximum likelihood estimation requires a statistical model of the system to which it is being applied. Scattering events are random occurrences that happen sparsely in time and very few photons arrive at a single super-pixel, even over tens of thousands of acquisitions. Such events are well modelled by the Poisson distribution, given by

$$P(X = x) = \frac{e^{-\mu} \mu^x}{x!},$$  \hspace{1cm} (4)

where \( X \) is the random variable and \( \mu \) is the characteristic parameter of the distribution, which for this case is the average number of photons arriving during each acquisition. For Poisson processes \( \mu \) is both the mean and the variance of the distribution.

Since there are so few photon arrivals most acquisitions contain only read noise (no photons). When the individual empty acquisitions are compiled into a histogram the result is a normal distribution centered on the mean value of the noise, \( \mu_N \) (in counts), for that super-pixel (see Figure 2). When a photon does arrive at the detector it will produce a count value given by \( \mu_N + k\beta \), where \( k \) is the number of photons arriving during the acquisition time and \( \beta \) is the mean camera gain in counts per photon. There will be some small variation in the count level produced by any given photon due to noise present in the amplification process (beta noise), and noise from sources in the detector (dark noise, read noise, etc.). For the system described the mean value of \( \beta \) is 38, and the standard deviation of the noise in \( \beta \) is approximately 4. Identifying photon arrivals is relatively simple because the noise distribution is much narrower than the photon gain. This is not always the case, and the noise sources must be carefully characterized in any system in order to understand sources of error that may be induced by using the method described in this paper (such as greater detector noise, pixel cross-over, photon pile-up, etc.).

Using knowledge of the noise parameters for the system and the nature of photon arrival the data can be processed in a way that essentially performs advanced photon counting, where multiple photon arrivals are counted accurately. First, the distribution is shifted by subtracting \( \mu_N \) from the count values such that the mean of the count values produced by non-events is zero. Second, all values are divided by the detector gain, \( \beta \), with the goal of transforming the data from counts into the number of photons that produced that count value. A sample histogram of the data after these operations have been performed can be seen in Figure 3. Because there is beta noise associated with the amplification and also noise

![FIG. 2. Histogram of the number of counts detected at a super-pixel during emission spectrum acquisition with an exposure time of 20 ns. This figure demonstrates that most acquisitions contain only read noise centered on the mean of the noise \( \mu_N \), which is 85 counts per acquisition for this particular super-pixel.](image)

![FIG. 3. Histogram of emission data after being shifted by \( \mu_N \) and divided by \( \beta \).](image)
from other sources the data values do not transform exactly to integer values (e.g., it appears as if fractional photons are arriving). The data are then rounded to the nearest integer value, the result of which can be seen in Figure 4. Justification of the rounding process can be found in Sec. V.

When the data are processed in this manner the resulting histogram is very similar to a Poisson distribution (described previously) with a very small characteristic parameter \( \mu \) that corresponds to the mean photon arrival rate. If one were to generate a Poisson random variable to simulate photon arrivals with \( \mu \ll 1 \) and sample it 27,000 times, the result would be mostly zeros corresponding to non-events and very few arrivals corresponding to one or more photons. The transformed data follow this distribution reasonably well (see Figure 4) and can be fit using a maximum likelihood estimation algorithm with a Poisson model. The resulting maximum likelihood estimator, \( \hat{\mu} \), provides the most likely arrival rate for the given data, which includes multiple photon events. For the processed data shown in Figure 4 the calculated value of \( \hat{\mu} \) is 0.0014 photons per acquisition.

C. MLE processing algorithm

Processing of the laser Thomson scattering data is performed using a multi-step algorithm. As described in Sec. III, the iCCD program stores each individual acquisition as a separate ascii file. Prior to performing plasma measurements, noise measurements are performed with the spectrograph entrance slit closed. Using these measurements the mean (\( \mu_N \)) and standard deviation (\( \sigma_N \)) of the noise are calculated for each individual super-pixel and are stored in new variables.

Each acquisition for the plasma measurements and the Rayleigh calibration measurements are also stored in individual ascii files. These files are loaded into Matlab\( ^\circ \) where they are combined into 3D matrices containing all of the individual acquisitions. After this pre-processing step there are three matrices containing the total spectrum measurements, the emission spectrum measurements, and the Rayleigh spectrum measurements (for calibration purposes, described later).

The algorithm used to process the data is as follows:

1. A single super-pixel is selected and a vector containing the count values for all acquisitions is created from the master data matrix.
2. The mean value of the noise at this super-pixel, \( \mu_N \) as determined from the pre-recorded noise measurements, is then subtracted from all of the values in order to shift the noise to zero mean.
3. After shifting, all values are divided by the mean detector gain \( \beta \).
4. All values are then rounded to the nearest integer, which combined with the previous step converts the data from counts to photons.
5. This corrected vector is provided to Matlab’s maximum likelihood estimation function and fit with a Poisson distribution. The maximum likelihood estimator, \( \hat{\mu} \), corresponds to the mean photon arrival rate at that super-pixel.

This process was performed on every super-pixel two times, once for each of the spectra sequences measured (total and emission). The results of this process are two-dimensional matrices the size of the binned detector that contain the estimated photon arrival for each data set. The corrected spectrum is constructed by simply subtracting the emission spectrum from the total spectrum and applying a mask in software to the stray/Rayleigh component near the laser wavelength.

Typically a stray light measurement is performed with the laser on and the discharge off,\(^ {18} \) and this spectrum is also subtracted from the total spectrum to create the final corrected Thomson spectrum. A set of stray light measurements was performed at the beginning of this test before the thruster was turned on, and since the stray light does not depend on the discharge conditions, additional measurements were not performed at each operating condition. After the test it was discovered that a coating was deposited on the lens during thruster operation, and since the coating was not present during the initial stray light measurements, the unattenuated stray light spectrum could not be subtracted from the attenuated total and emission spectra.

By not subtracting the stray light contribution it is possible that the spectral shape and amplitude were affected. The triple spectrograph has excellent stray light redistribution of \( 8 \times 10^{-3} \) at \( \Delta \lambda = 1 \) nm from the laser wavelength, and decreases rapidly to \( \approx 5 \times 10^{-6} \) for \( \Delta \lambda > 3 \) nm. The stray light level for the scattering system used in this work was equivalent to Rayleigh scattering on N\(_2\) gas with a density of 2.04 \( \times 10^{22} \) m\(^{-3}\). A velocity distribution must be assumed for the electrons and for convenience we will assume that the electrons in the plasma are thermalized, implying that the Thomson spectrum is Gaussian in shape. The ratio of the Thomson scattered power to the redistributed light power can then be expressed as\(^ {26} \)

\[
\frac{P_T}{P_R} = 0.3 \frac{n_e}{n_{N_2}} \frac{d\sigma_T/d\Omega}{d\sigma_R/d\Omega},
\]  

(5)
where \( R \) is the redistribution level of the system (units of nm\(^{-1}\)), \( n_e \) is the electron number density, \( n_{N_2} \) is the calibration gas number density (nitrogen), and \( d\sigma_T/d\Omega \) and \( d\sigma_R/d\Omega \) are the differential Thomson and Rayleigh scattering cross-sections, respectively. We can now calculate the Thomson to stray light ratio for a range of plasma conditions and see how much effect the stray light will have. To begin, we assume an electron density of \( 1 \times 10^{17} \) m\(^{-3} \) and an electron temperature of 1 eV. Using these values we can calculate that the \( \Delta\lambda_{1/e} \) half-width for a temperature of 1 eV is approximately 1.5 nm. The stray light redistribution level at \( \Delta\lambda = 1.5 \) nm is approximately \( 4.5 \times 10^{-5} \), indicating a Thomson to stray light ratio of 4.27. This level is not tremendous, but clearly sufficient to detect the Thomson signal. Previous probe measurements of electron temperature in the near-field plume of the thruster used in this work indicated temperatures of approximately 4.5 eV, and if we use this new temperature value we find that the Thomson to stray light ratio increases to 24. If we also increase the electron density to \( 5 \times 10^{17} \) m\(^{-3} \) we find that the ratio increases to over 120, and the stray light contribution to the Thomson spectrum is negligible.

We have established that for electron temperatures above 1 eV and electron densities above \( 1 \times 10^{17} \) m\(^{-3} \) the contribution of stray light is low to negligible. In general, the higher the temperature and density the less the redistributed stray light matters. Since it was not possible to accurately subtract the stray light contribution from the total spectra obtained in this work, the corrected Thomson spectra were determined by subtracting the emission signal from the total signal and simply masking the center of the spectrum which is contaminated by stray light in software. The mask covers a 2 nm range (±1 nm from 532 nm), which covers the range at which the Thomson to stray light ratio is less than 2 for an electron temperature of 0.5 eV and electron density of \( 1 \times 10^{17} \) m\(^{-3} \). This temperature and density combination is considered to be a practical minimum for the system, and for higher temperatures and densities only a small fraction of the usable portion of the Thomson spectrum is excluded from data processing.

D. Determination of electron temperature and density

Although in principle no assumptions are needed regarding the electron distribution function, the low photon arrival rate present in this experiment yields a low signal-to-noise ratio (SNR). Small deviations from a Maxwellian distribution are difficult to detect in this case, and without high SNR measurements indicating otherwise a Maxwellian distribution was assumed. Using this assumption a Gaussian fit can be performed and the conventional thermal temperature can be used to characterize the distribution. A nonlinear least-squares Gaussian fit was performed on the corrected scattering spectrum and the value of \( \sigma \) determined from the fit was used to calculate the 1/e width according to the following equation:

\[
\Delta\lambda_{1/e} = \sqrt{2}\sigma^2.
\]  

This value was then substituted into the following equation for the electron temperature:

\[
T_e = \frac{m_e e^2}{8 k_B \sin^2(\theta/2)} \left( \frac{\Delta\lambda_{1/e}}{\lambda_i} \right)^2.
\]  

An absolute measurement of density can be obtained by performing Rayleigh calibration on a gas of known pressure. The density is determined according to the following equation:

\[
n_e = n_R \frac{\sigma_R}{\sigma_T} \frac{P_T}{P_R} \frac{\int \lambda R}{\int \lambda T}.
\]  

where \( n_R \) is the number density of the calibration gas (for this work \( N_2 \) was used), \( \frac{P_T}{P_R} \) is the ratio of laser powers for the two measurements, and \( \int \lambda R \) is the integrated intensity of each spectrum. The cross-sections for Thomson and Rayleigh scattering (\( \sigma_T \) and \( \sigma_R \)) are known constants and the number density during Rayleigh calibration (\( n_R \)) can be calculated using the ideal gas law. The integral of the Rayleigh spectrum is performed numerically, and for the Thomson spectrum the integral is calculated from the amplitude and standard deviation of the fit to the corrected spectrum.

Rayleigh calibration was performed prior to pumping down the vacuum chamber to hard vacuum. The chamber was roughed down to 2.5 Torr and the left overnight to allow dust to settle. The laser was then fired into the vacuum chamber and the camera recorded the scattered light with the same settings that were used for laser measurements (27,000 acquisitions, 20 ns exposure). After obtaining the Rayleigh calibration data the chamber was pumped down to hard vacuum overnight and plasma measurements were performed over the next few days.

V. RESULTS

An example of a corrected spectrum and corresponding fit can be seen in Figure 5. The spectrum shown in this figure contains significant noise and it is necessary to consider the accuracy of the electron temperature and density values.
determined from the spectrum. When using pulse accumulation methods and photon counting methods with PMTs the process of defining a signal to noise ratio is well defined, but after transforming the data using the MLE processing method a SNR is difficult to define.

The first of two key sources of uncertainty in the MLE processed laser measurements is the uncertainty in the mean photon arrival rate at a given wavelength. Uncertainty in the mean photon arrival rate at a super-pixel arises due to the noise present in the detected number of counts and the rounding that occurs during MLE processing. The two specific occurrences that contribute to this uncertainty are the recording of a noise-generated count value as a real photon arrival and attributing a count value generated by a real photon arrival event to the wrong number of photons (for example, counting the arrival of two photons as a single photon). In the first situation the probability of counting a noise value as a real photon event is quite small. Since the standard deviation of the noise is small compared to the camera gain on average only one noise-generated count value is incorrectly attributed to an actual photon arrival during a 27,000 shot sequence. This miscount rate corresponds to a mean photon arrival rate of 0.000037, which corresponds to 10% or less of the mean photon arrival rate at a super-pixel half-width for the spectra that have been measured using this diagnostic system (see Figure 6).

The second source of uncertainty in the mean arrival rate is the miscounting of real photon arrivals. The count levels that are generated by real photon arrivals contain shot noise on the signal itself as well as noise due to the amplification process. The shot noise for an iCCD is given by

\[ \sigma_{\text{shot}} = \beta \times F \times \sqrt{\eta \phi_p \tau}, \]

where \( \beta \) is the camera gain (38), \( F \) is the noise factor (due to the gain process), \( \eta \) is the quantum efficiency, \( \phi_p \) is the photon flux in photons per second per super-pixel, and \( \tau \) is the acquisition time. Converting from photons to counts and substituting the camera noise factor of 1.6 yield:

\[ \sigma_{\text{shot}} = 1.6 \times \sqrt{\text{Counts}}. \]

Since the noise level is a function of the count level, the count level influences the chances of a miscount occurring. The shot noise and detector noise are uncorrelated and add in quadrature, such that

\[ \sigma_{\text{total}} = \sqrt{\sigma_{\text{shot}}^2 + \sigma_{\text{detector}}^2}. \]

Using 38 as the mean count level for single photon arrivals the rate at which a single photon arrival will be counted as a 0 or 2 is approximately 8% (based on the noise statistics). For two-photon arrivals using 76 as the mean count value yields a miscount rate of approximately 20%. The miscount rate continues to increase for higher count values, but the signal to noise ratio actually increases since the error from counting a single photon as zero or two photons is much greater than the error from counting three photons as two or four photons. Since most of the arrivals are single photon events with a only a few multi-photon events the uncertainty in the mean arrival rate determined from these calculations can be estimated at approximately 10%–15%.

The second key source of uncertainty in the MLE processed laser measurements is the uncertainty in the fit of the scattered spectrum. The first and most obvious is the assumption that the plasma is Maxwellian. If a Maxwellian plasma is assumed it is important to consider the certainty that can be placed on the determination of the fit parameters of the scattered spectrum. Using the residuals calculated from the nonlinear least-squares fit to the spectrum it is possible to compute 95% confidence intervals for the fit parameters. While these confidence intervals are not predictive, reasonable bounds can be put on the estimates of electron temperature and density calculated from the scattered spectra. An example of these bounds for a good spectrum can be seen in Figure 6. The calculated values of temperature and density at this operating condition for the best fit are 5.93 eV and 6.96 \( \times 10^{17} \text{ m}^{-3} \), respectively. If the parameter estimates corresponding to the confidence interval bounds are used to calculate the electron temperature and density the upper values provide a temperature of 8.98 eV and a density of 1.09 \( \times 10^{18} \text{ m}^{-3} \), which are approximately 50% higher than the values given by the best fit to the data. The lower values yield a temperature of 3.52 eV and a density of 3.91 \( \times 10^{17} \text{ m}^{-3} \), which are approximately 40% lower than the values calculated from the best fit.

The effect of the curve fit to the spectrum is more pronounced when the Thomson signal is weaker. There are multiple reasons for a weaker signal, including a coating on the collection lens, lower electron density, and misalignment of the laser and collection volume. Alignment of the probing laser and the collection lens axis has a strong effect on the signal in scattering measurements. The Thomson signal is produced by the laser beam and if the optics are not capturing the scattered photons no signal is measured at the detector. On the other hand, xenon emission is present nearly everywhere in the plasma, and even if the collection optics are not targeted near the laser beam a strong emission signal will be detected. The mean value of the emission signal can be subtracted but the noise on this signal cannot, making the fit less certain. An example of the fit to a spectrum with less detected Thomson
signal can be seen in Figure 7. The calculated values of temperature and density at this operating condition for the best fit are 9.48 eV and 3.56 $\times 10^{17}$ m$^{-3}$, respectively. The upper confidence interval values provide a temperature of 20.7 eV and a density of 6.80 $\times 10^{17}$ m$^{-3}$, which are approximately 120% and 90% higher (respectively) than the values given by the best fit to the data. The lower values yield a temperature of 2.82 eV and a density of 1.39 $\times 10^{17}$ m$^{-3}$, which are approximately 70% and 60% lower (respectively) than the values calculated from the best fit.

Given the sources of uncertainty just discussed and the uncertainty calculated from the fits to the data, the uncertainty of the MLE processed laser measurements presented in this work is estimated at 30%–100% depending on the scattered signal strength. The uncertainty in the fit parameters is a more dominant factor than the uncertainty in the mean arrival rate at a given wavelength, and decreases with increased Thomson signal. In addition to the uncertainty in the fit of the Thomson spectrum, determination of the Rayleigh scattering spectrum that is used to calibrate the density measurements can affect the density estimates. Although the same camera and spectrograph settings are used during measurement of the Rayleigh signal as for the scattered signal, the Rayleigh signal is in a different photon flux regime than the Thomson signal. There is greater certainty in the estimation of the Rayleigh scattering spectrum than the Thomson spectrum due to higher SNR, but the difference in relative uncertainty limits the overall confidence in the density estimates provided by the MLE method.

**VI. SUMMARY**

A new maximum likelihood estimation-based data processing method has been developed for application to laser Thomson scattering data in situations where the scattered photon flux is very low. This method is applicable in any situation where the number of detected photons is low, whether due to low electron density or poor optical access/small detection solid angle. Use of an iCCD allows measurement of the entire scattered spectrum at once, which decreases acquisition times or improves the detection limit for a fixed measurement time. Use of this method can provide two key improvements over photon counting methods. The first (and arguably most important) improvement is that the statistically significant non-arrival events are included in the data processing algorithm. The second improvement is that the new algorithm can accurately handle both single and multi-photon events using the equipment described in this work. The iCCD exhibited negligible blooming, less than 0.005% smear, and a high gain to noise ratio. Other systems may not handle multi-photon events as well using this algorithm.

This new maximum likelihood estimation-based method was applied to laser Thomson scattering measurements performed in the near-field plume of a 2-kW Hall-effect thruster. The low plasma density in the thruster plume produced low SNR measurements of the scattered spectrum, and there was significant uncertainty in the determination of electron temperature and density. Even so, reasonable measurements of electron temperature and density were obtained using the new data processing method. Improvements in detector quantum efficiency, stray light reduction, and total system throughput will increase certainty in the laser measurements and provide more accurate measurements of electron temperature and density in the near-field plume. Although the new processing algorithm was effective at very low photon arrival rates (less than 1 photon per super-pixel per 1000 laser pulses on average), it is applicable to a wide range of photon arrival rates. As long as a model for the collected scattering can be supplied to the maximum likelihood estimation algorithm the processing method described in this paper can be applied to a wide range of plasmas.

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