MEEM 3700
Mechanical Vibrations

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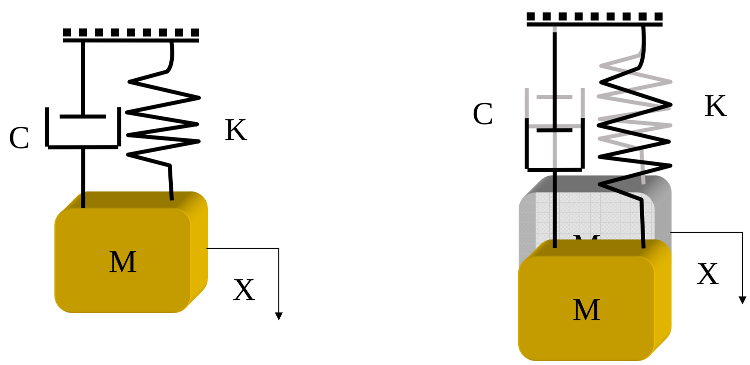
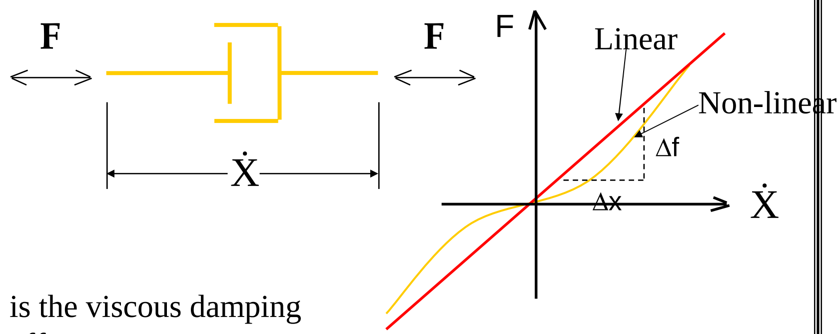
Single Degree of Freedom Damped Free Vibration



Given an initial condition, Determine the resulting motion

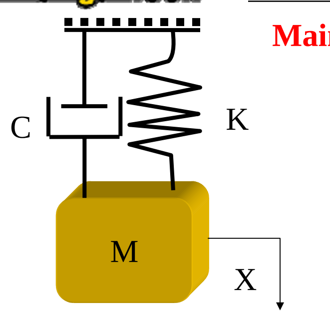
Viscous Damping Element (Dashpot)

- Damping Force is Linear and Proportional to Velocity.



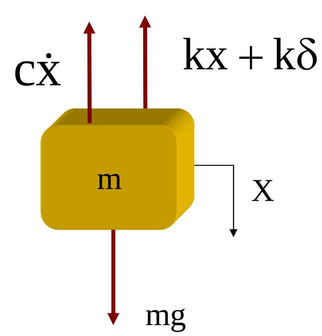
C is the viscous damping coefficient
units: N-sec/m or lbf-sec/ft

$$F = C\dot{X}$$



Maintain Dynamic Equilibrium

Free Body Diagram



At rest, X= 0 (static equilibrium)
mg = kδ

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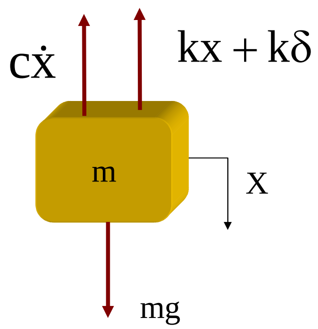
Free Body Diagram

Maintain Dynamic Equilibrium



Apply Newton’s 2nd Law

$$\sum F = m\ddot{x}$$

$$\sum F_{x\downarrow +} = m\ddot{x}$$

$$mg - (kx + k\delta) - c\dot{x} = m\ddot{x}$$

Equation of motion $$m\ddot{x} + c\dot{x} + kx = 0$$

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Equation of motion $$m\ddot{x} + c\dot{x} + kx = 0$$

2nd order Differential equation

homogeneous

linear

Constant coefficients

Form of solution:

$$x(t) = X\sin(\omega t + \Phi) \quad \text{or} \quad x(t) = Ce^{st}$$

Equation of motion $m\ddot{x} + c\dot{x} + kx = 0$

Assume $x(t) = Ce^{st}$

$$\dot{x}(t) = sCe^{st}$$
$$\ddot{x}(t) = s^2Ce^{st}$$

$$ms^2Ce^{st} + scCe^{st} + kCe^{st} = 0$$

$$(ms^2 + cs + k)Ce^{st} = 0$$

for a non-trivial solution

$$ms^2 + cs + k = 0$$

Equation of motion $m\ddot{x} + c\dot{x} + kx = 0$

$$ms^2 + cs + k = 0$$

$$s_{1,2} = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

if s_1 and s_2 are not equal

C_1 and C_2 are determined from initial conditions

$$s_{1,2} = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

Consider a case when, $c^2 - 4mk = 0$

Solving for c:

$c = 2\sqrt{km} = C_c$

C_c = critical damping

Define:

$\omega_n = \sqrt{\frac{k}{m}}$ = natural frequency

$\zeta = \frac{c}{C_c}$ = damping ratio

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta^2 - 1)}\omega_n$$

Case 1: $\zeta < 1$ Under damped (Complex conjugate roots)

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta^2 - 1)}\omega_n$$

$$s_{1,2} = -\zeta\omega_n \pm j\sqrt{(1-\zeta^2)}\omega_n$$

Define:

$\omega_d = \sqrt{(1-\zeta^2)}\omega_n$ = damped natural frequency

$$s_{1,2} = -\zeta\omega_n \pm j\omega_d$$

Case 1: $\zeta < 1$ Under damped (Complex conjugate roots)

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

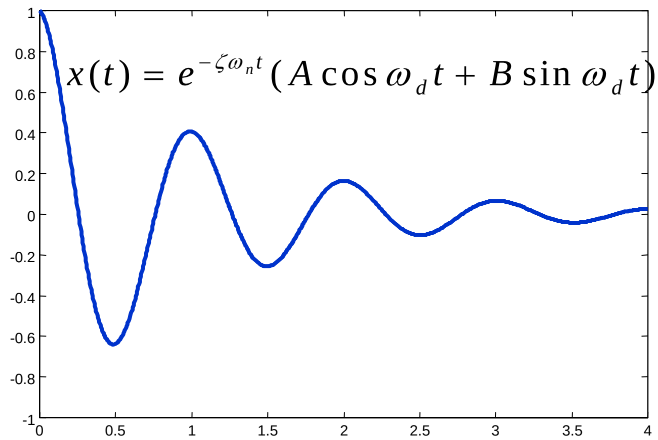
$$x(t) = C_1 e^{(-\zeta\omega_n + j\omega_d)t} + C_2 e^{(-\zeta\omega_n - j\omega_d)t}$$

This simplifies to:

$$x(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

where A & B are arbitrary constants to be found from initial conditions

Case 1: $\zeta < 1$ Under damped (Plot of x(t) vs. time)



Case 2: $\zeta = 1$ Critically damped (Real equal roots)

$$s_{1,2} = -\zeta\omega_n$$

$$s_1 = -\omega_n$$

$$s_2 = -\omega_n$$

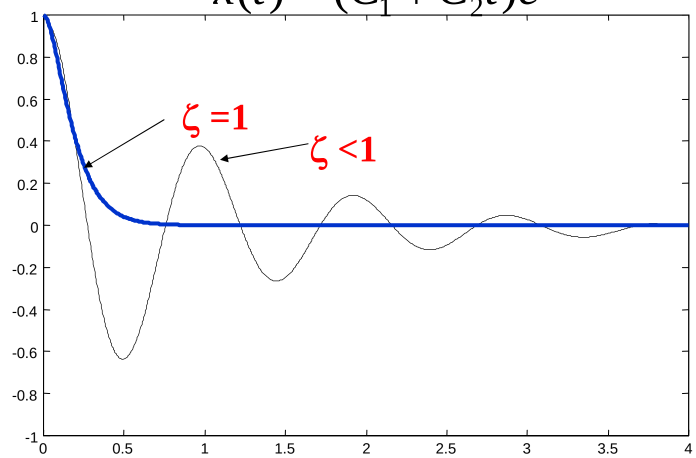
$$x(t) = C_1 e^{s_1 t} + C_2 t e^{s_2 t} \quad \text{or}$$

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

C_1 and C_2 are constants to be found from initial conditions

Case 2: $\zeta = 1$ Critically damped (Real equal roots)

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$



Case 3: $\zeta > 1$ Over-damped (Real unequal roots)

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta^2 - 1)}\omega_n$$

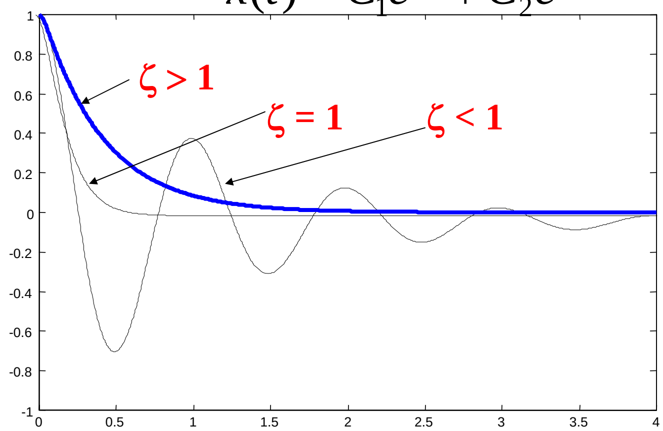
$$s_1 = -\zeta\omega_n + \sqrt{(\zeta^2 - 1)}\omega_n$$

$$s_2 = -\zeta\omega_n - \sqrt{(\zeta^2 - 1)}\omega_n$$

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

Case 3: $\zeta > 1$ Over-damped (Real unequal roots)

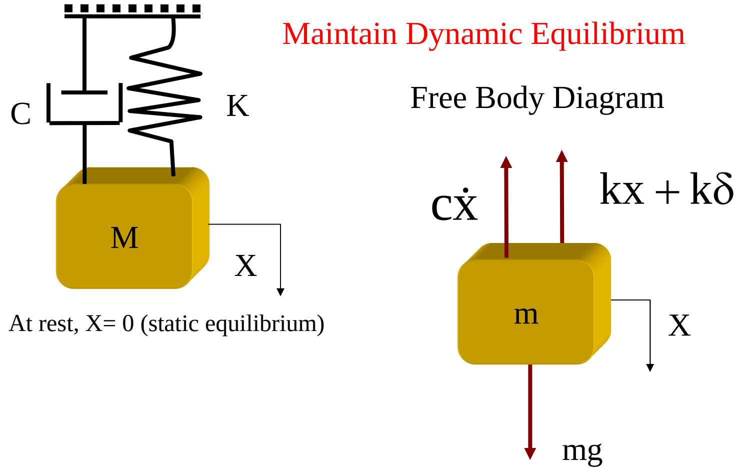
$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$



Review

Maintain Dynamic Equilibrium

Free Body Diagram



At rest, X= 0 (static equilibrium)

$$m\ddot{x} + c\dot{x} + kx = 0$$

Case 1: $\zeta < 1$ Under-damped (Complex conjugate roots)

$$x(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

Or alternatively, we can also write

$$x(t) = e^{-\zeta\omega_n t}\left(X\,\sin\left(\omega_d t + \phi\right)\right)$$

$$where\ X = \sqrt{A^2 + B^2}\ \text{ and }\ \phi = \tan^{-1} A/B$$

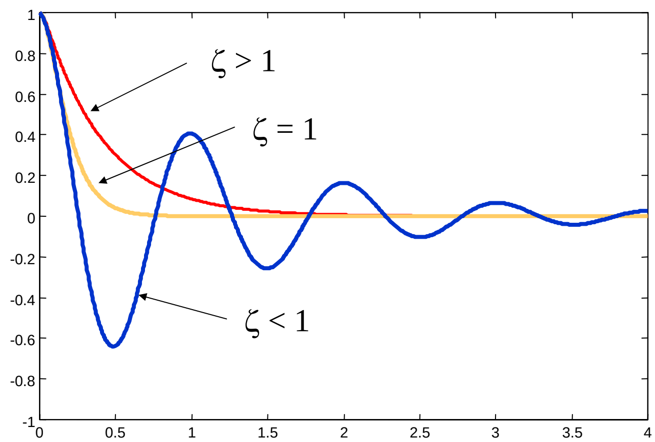
Case 2: $\zeta = 1$ Critically damped (Real equal roots)

$$x(t) = (C_1 + C_2 t)e^{-\zeta\omega_n t}$$

Case 3: $\zeta > 1$ Over-damped (Real unequal roots)

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \qquad s_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta^2 - 1)}\omega_n$$

Free Vibration Response: SDOF with Damping



Damping ratio (for many structural materials) $0.001 \le \zeta \le 0.05$

% critical damping $\zeta * 100\%$

$$0.1\% \le \zeta \le 5\%$$

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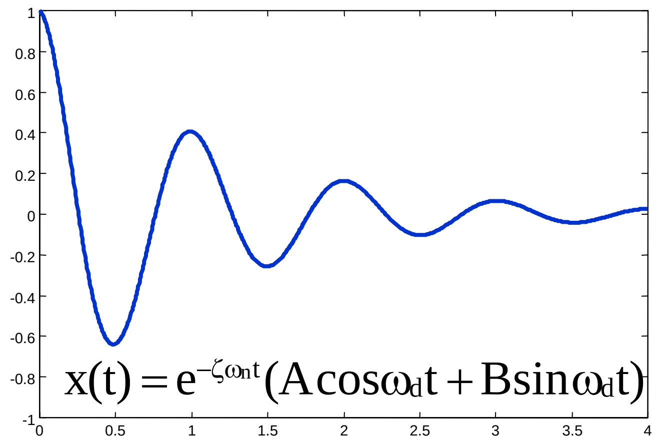
Case 1: $\zeta < 1$ Under damped (Complex conjugate roots)



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$$x(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

Determine A, and B from the given initial Conditions, x(0) and $\dot{x}(0)$

$$x(0) = \overset{1}{e^{-\zeta\omega_n 0}}(A\cos(\overset{1}{\omega_d 0}) + B\sin(\overset{0}{\omega_d 0}))$$

$$x(0) = A$$

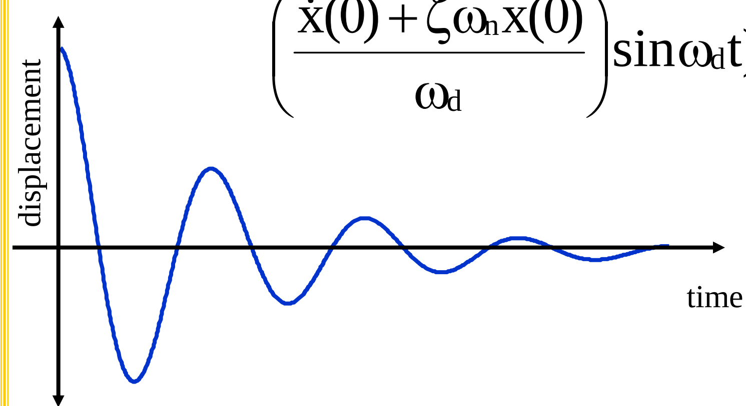
$$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}(x(0)\cos\omega_d t + B\sin\omega_d t) + e^{-\zeta\omega_n t}(-\omega_d x(0)\sin\omega_d t + \omega_d B\cos\omega_d t)$$

$$\dot{x}(0) = -\zeta\omega_n e^{-\zeta\omega_n 0}(x(0)\cos(\omega_d 0) + B\sin(\omega_d 0)) + e^{-\zeta\omega_n t}(-\omega_d x(0)\sin(\omega_d 0) + \omega_d B\cos(\omega_d 0))$$

$$\dot{x}(0) = -\zeta\omega_n x(0) + \omega_d B$$

$$B = \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}$$

$$x(t) = e^{-\zeta\omega_n t}\left(x(0)\cos\omega_d t + \left(\frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}\right)\sin\omega_d t\right)$$



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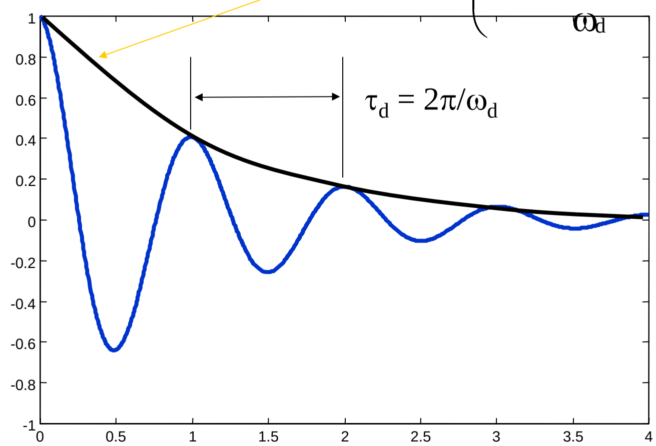
$$x(t) = e^{-\zeta\omega_n t}\left(x(0)\cos\omega_d t + \left(\frac{\dot{x}(0)+\zeta\omega_n x(0)}{\omega_d}\right)\sin\omega_d t\right)$$



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Alternate form of Equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0$$

$\omega_n = \sqrt{\frac{k}{m}} =$ natural frequency $\qquad \zeta = \frac{c}{C_c} =$ damping ratio

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}$$