MEEM 3700
Mechanical Vibrations

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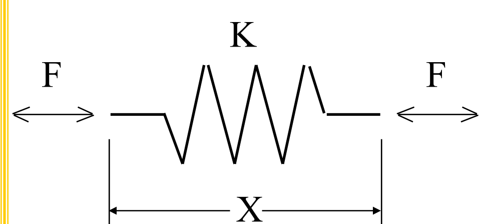
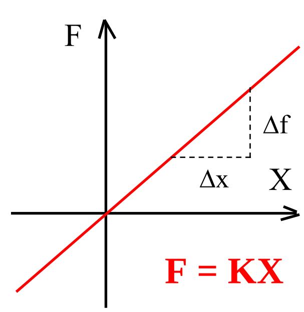
Sections 1.7 & 1.8 in the text

Stiffness Element: Spring

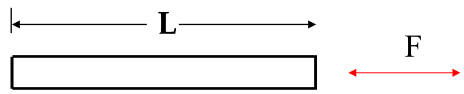
- Spring Force is Linear and Proportional to Displacement.





F = KX

Stiffness Elements: Bar in Tension/Compression

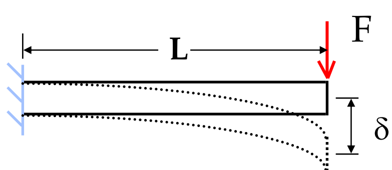


$$K = \frac{AE}{L}$$

A= Area of Cross-Section
E= Modulus of Elasticity
L= Length

Stiffness Element: Beams in Bending

- E = modulus of elasticity.
- I = area moment of inertia relative to bending axis.
- K_b = spring rate of beam.



$$F = \frac{3EI}{L^3}\delta$$

$$\delta = \frac{FL^3}{3EI}$$ for a cantilever beam

$$K_b = \frac{3EI}{L^3}$$

Similarly obtain K_b for other boundary conditions from deflection equations

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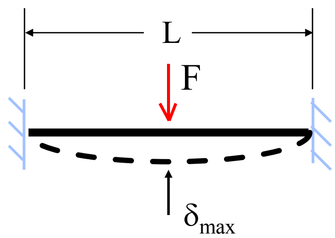
Stiffness Element: Beams in Bending

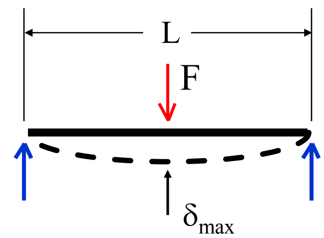
Fixed-Fixed



$F = K\delta_{max}$ $\quad \delta_{max} = \dfrac{FL^3}{192EI}$

$$K_b = \frac{192EI}{L^3}$$

Simply Supported



$F = K\delta_{max}$ $\quad \delta_{max} = \dfrac{FL^3}{48EI}$

$$K_b = \frac{48EI}{L^3}$$

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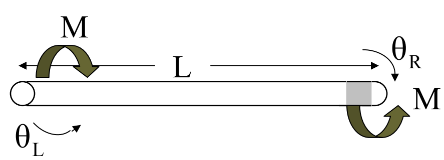
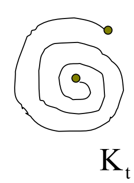
Stiffness Element: Torsion





M=Moment

$$M = K_t\left|(\theta_L - \theta_R)\right| = K_t\Delta\theta$$

$$= K_t\theta$$

$K_t = \dfrac{GJ}{L}$ $\quad J = \pi d^4 / 32$

for a round shaft of diameter d

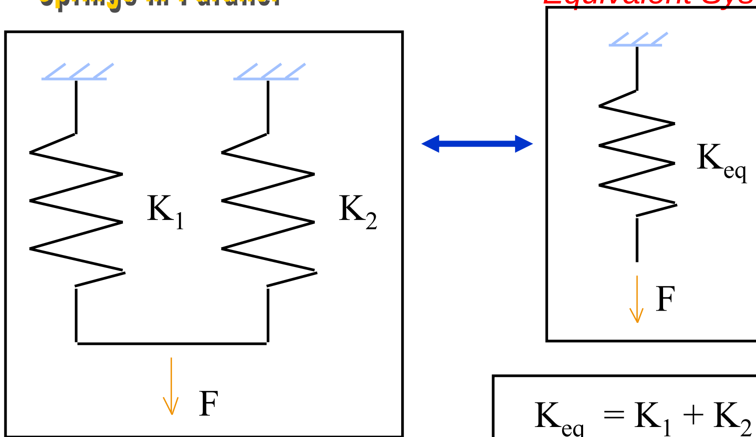
(J = Polar area moment)

$$\text{units} = \frac{N \times m}{rad}$$

How to combine Stiffness Elements?

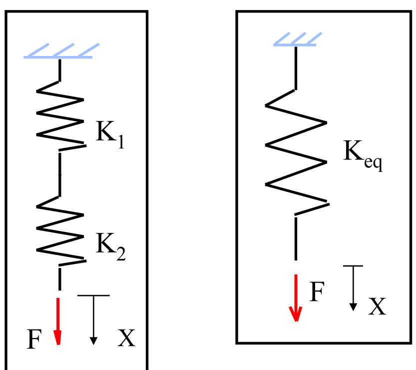
Springs in Parallel

Equivalent System

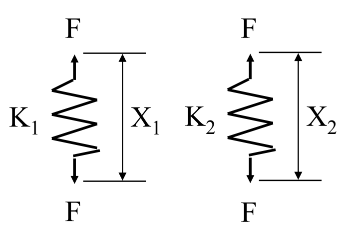


$$K_{eq} = K_1 + K_2$$

Springs in Series



Equivalent System



$$X = X_1 + X_2$$

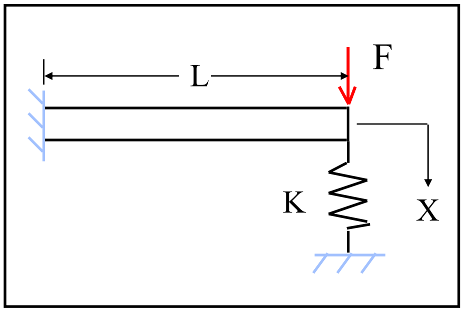
$$F = K_{eq}X = K_1X_1 = K_2X_2$$

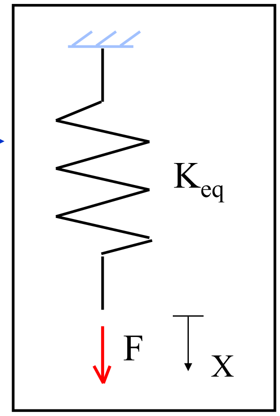
$$X = \frac{K_{eq}X}{K_1} + \frac{K_{eq}X}{K_2}$$

$$\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2}$$

$$K_{eq} = \frac{K_1K_2}{(K_1 + K_2)}$$

Spring in Parallel

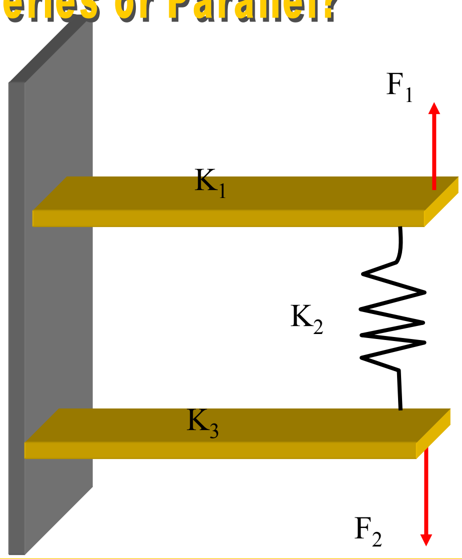




$$K_{eq} = K + \frac{3EI}{L^3}$$

Equivalent System

Series or Parallel?



Case 1

$$F_1 = K_{eq} X$$

$$K_{eq} = K_1 + \frac{K_2 K_3}{K_2 + K_3}$$

Case 2

$$F_2 = K_{eq} X$$

$$K_{eq} = K_3 + \frac{K_2 K_1}{K_2 + K_1}$$

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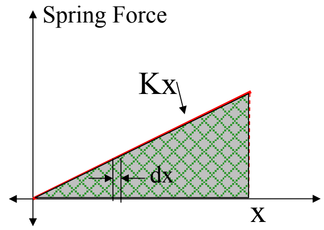
Energy Concepts: Stiffness Elements

U = Potential Energy

$$U = \frac{1}{2}Kx^2$$

or $U = \frac{1}{2}K_t\theta^2$

for a torsional spring



$$U = Kxdx$$

$$U = \int_0^x Kxdx = \frac{1}{2}Kx^2$$

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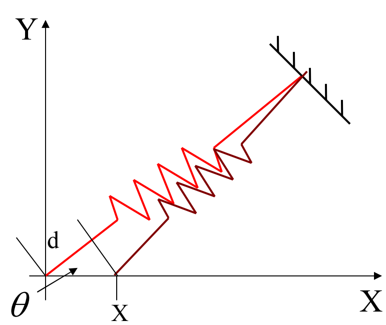
Stiffness Element: Spring at an Angle



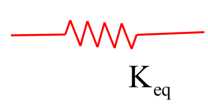
$$d = x\cos\theta$$

Equivalent potential energy

$$U = \frac{1}{2}k\left(x\cos\theta\right)^2 = \frac{1}{2}k_{eq}x^2$$

$K_{eq} = K\cos^2\theta$

in the x-direction

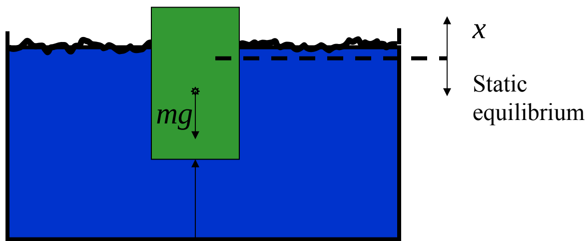


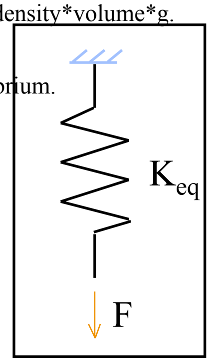
Stiffness Element: Buoyancy

Archimedes' principle: The buoyant force acting on a floating body is equal to the weight of the liquid displaced by the body.

Spring Force (F)= Weight of the fluid displaced = mass*g =density*volume*g.
Volume displaced= cross-sectional area*displacement *x*.

Hence, F= ρgA*x*= K_{eq} *x*, x is measured from the static equilibrium.

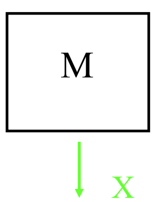


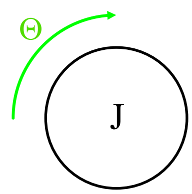


$$F_b = mg + \rho g A x$$

K_{eq}=ρgA

Mass/Inertia Elements





J = mass moment of inertia $(kg \times m^2)$

Mass

Rotating Inertia

Rigid Body Behavior

Gain / Lose Kinetic Energy

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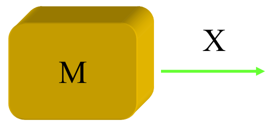
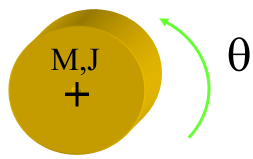
Mass/Inertia Elements

Kinetic Energy of translation



$$K.E. = \frac{1}{2} m\dot{x}^2$$

Kinetic Energy of rotation



$$K.E. = \frac{1}{2} J\dot{\theta}^2$$

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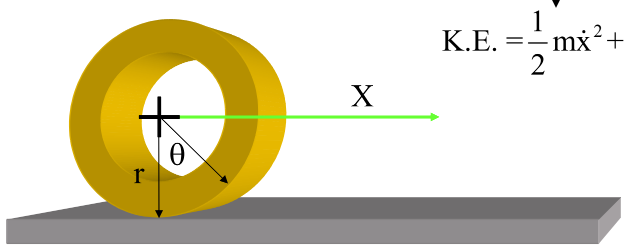
Mass/Inertia Elements

Kinetic Energy of rolling motion

translation rotation

$$K.E. = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J\dot{\theta}^2$$



NOTE: $x = r\theta$ therefore $K.E. = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J\frac{\dot{x}^2}{r^2}$

$K.E. = \frac{1}{2} m_{eq}\dot{x}^2$ where $m_{eq} = m + J/r^2$

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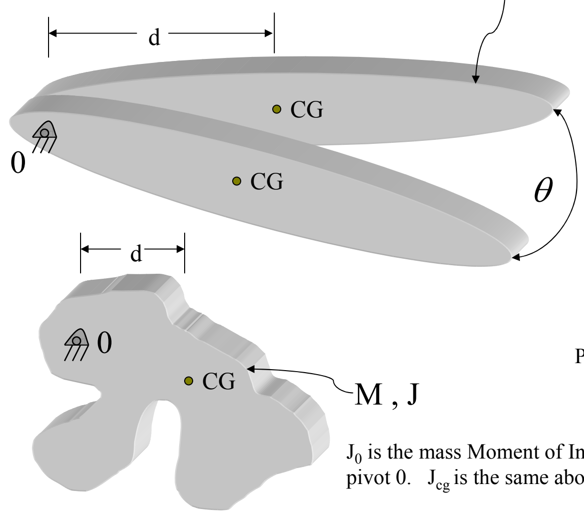
Mass/Inertia Elements



$$K.E. = \frac{1}{2} J_0 \dot{\theta}^2$$

Parallel Axis Theorem (PAT)

$$J_0 = J_{cg} + Md^2$$

J_0 is the mass Moment of Inertia of the body about the pivot 0. J_{cg} is the same about the center of mass

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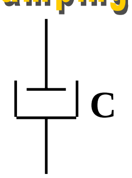
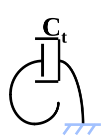
Damping Elements

Viscous damping



$$F = C\dot{x}$$

Linear Viscous Damper



$$M = C_t \dot{\theta}$$

Linear Torsional Viscous Damper