MEEM 3700
Mechanical Vibrations

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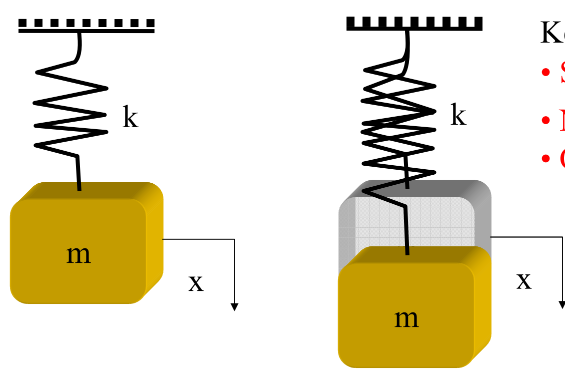
Single Degree of Freedom Free Vibration



Key Points:

- System is un-damped
- No external forces
- Only vertical motion

Given some initial conditions, Determine the resulting motion

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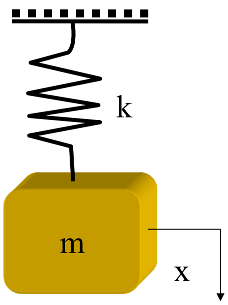
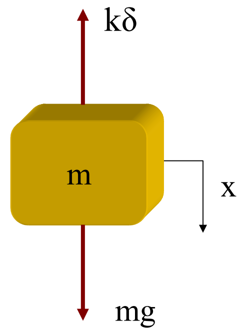
Free Body Diagram: Static Case



At rest, x = 0 (static equilibrium)



$$mg = k\delta$$

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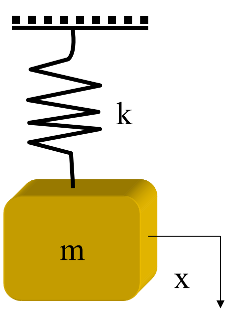
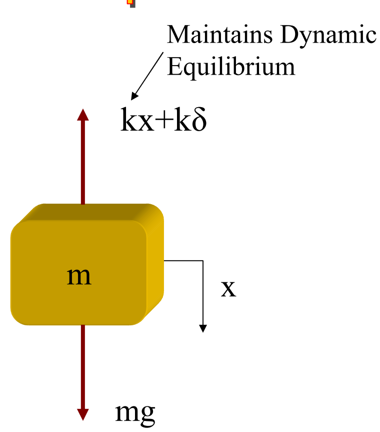
Free Body Diagram: Dynamic Equilibrium





Note: x is measured from the static equilibrium position.

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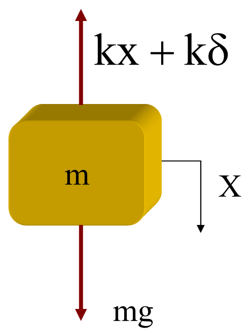
Free Body Diagram: Dynamic Case

Free Body Diagram



Apply Newton's 2nd Law

$$\sum F = m\ddot{x}$$

$$\sum F_{x\downarrow +} = m\ddot{x}$$

$$mg - (kx + k\delta) = m\ddot{x}$$

Equation of motion (EOM) $\boxed{m\ddot{x} + kx = 0}$

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Equation of motion $\boxed{m\ddot{x} + kx = 0}$

2nd order Differential equation

homogeneous

linear

constant coefficients

Forms of solution:

$x(t)=X \sin(\omega t + \Phi)$

$x(t)=X \cos(\omega t - \Phi)$

$x(t)=Ce^{st}$ ---We will use this form

Equation of motion $\boxed{m\ddot{x} + kx = 0}$

Assume $x(t) = Ce^{st}$ $\quad \dot{x}(t) = sCe^{st}$

$\ddot{x}(t) = s^2Ce^{st}$

$$ms^2Ce^{st} + kCe^{st} = 0$$

$$(ms^2 + k)Ce^{st} = 0$$

for a non - trivial solution

$$ms^2 + k = 0$$

Equation of motion $\boxed{m\,\ddot{x} + kx = 0}$

$$ms^2 + k = 0 \qquad s_{1,2} = \pm j\sqrt{\frac{k}{m}}$$

$$x(t) = C_1e^{s_1t} + C_2e^{s_2t}$$

$$\boxed{x(t) = C_1e^{j\sqrt{\frac{k}{m}}t} + C_2e^{-j\sqrt{\frac{k}{m}}t}}$$

C_1 and C_2 are arbitrary constants to be determined from initial conditions

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$$x(t) = C_1 e^{j\sqrt{\frac{k}{m}}t} + C_2 e^{-j\sqrt{\frac{k}{m}}t}$$

Recall Euler's identity: $e^{\pm j\theta} = \cos(\theta) \pm j\sin(\theta)$

$$x(t) = C_1\left(\cos\sqrt{\frac{k}{m}}t + j\sin\sqrt{\frac{k}{m}}t\right) + C_2\left(\cos\sqrt{\frac{k}{m}}t - j\sin\sqrt{\frac{k}{m}}t\right)$$

$$x(t) = (C_1 + C_2)\cos\sqrt{\frac{k}{m}}t + j\,(C_1 - C_2)\sin\sqrt{\frac{k}{m}}t$$

$$\boxed{x(t) = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t}$$

A & B are always real since C_1 and C_2 are Complex conjugates

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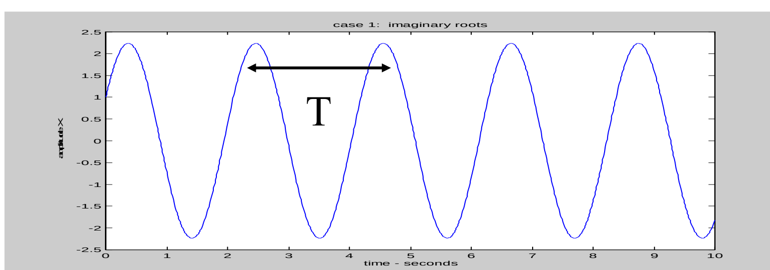
$$\boxed{x(t) = A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t}$$



$$\omega_n = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} = \text{natural frequency (rad/sec)}$$

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$$\boxed{x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)}$$

$$f_n = \frac{1}{T} = \text{natural frequency}\left(\frac{\text{cycles}}{\text{sec}}, \text{ or Hz}\right)$$

$$\boxed{x(t) = A\cos(2\pi f_n t) + B\sin(2\pi f_n t)}$$

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$$\boxed{x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)}$$

$x(0)$

$\dot{x}(0)$

A and B are determined from the **Initial Conditions**

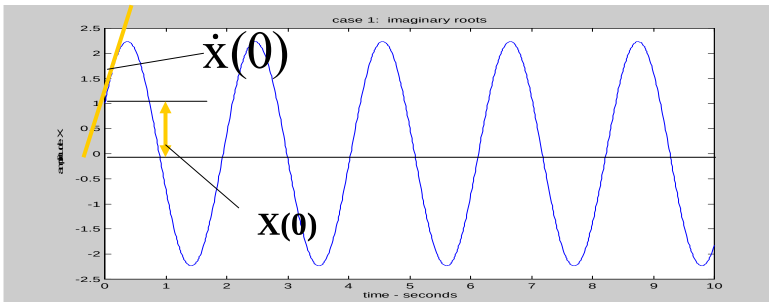
$$x(0) = A\cos(\underbrace{0}_{1}) + B\sin(\underbrace{0}_{0}) \qquad x(0) = A$$

$$\dot{x}(t) = -\omega_n A\sin(\omega_n t) + \omega_n B\cos(\omega_n t)$$

$$\dot{x}(0) = -\omega_n \cancel{A*(0)} + \omega_n B(1) \qquad \boxed{\frac{\dot{x}(0)}{\omega_n} = B}$$

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

$$\boxed{x(t) = x(0)\cos(\omega_n t) + \frac{\dot{x}(0)}{\omega_n}\sin(\omega_n t)}$$



Harmonic Motion

Different forms of the same solution:

$$x(t) = X\cos(\omega_n t - \phi)$$

$$X = \sqrt{A^2 + B^2} = \sqrt{x(0)^2 + \left(\frac{\dot{x}(0)}{\omega_n}\right)^2} \qquad \phi = \tan^{-1}\left(\frac{\dot{x}(0)}{x(0)\,\omega_n}\right)$$

Amplitude

Phase

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Harmonic Motion

Different forms of the same solution:

$$x(t) = X \sin\left(\omega_n t + \phi\right)$$

$$X = \sqrt{A^2 + B^2} = \sqrt{X(0)^2 + \left(\frac{\dot{X}(0)}{\omega_n}\right)^2} \qquad \phi = \tan^{-1}\left(\frac{x(0)\omega_n}{\dot{x}(0)}\right)$$

Amplitude

Phase

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Torsional System: SDOF

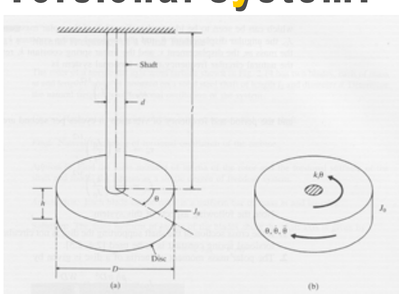

FIGURE 2.13 Torsional vibration of a disc.

Equation of Motion:

$$J_o\ddot{\theta} + k_t\theta = 0$$

Solution Form:

$$\theta(t) = A\cos\left(\omega_n t\right) + B\sin\left(\omega_n t\right)$$

Here $\omega_n = \sqrt{\frac{k_t}{J}} = \frac{2\pi}{T} =$ natural frequency (rad/sec)

Compare with Translatory System

Equation of Motion: $m\ddot{x} + k\ x = 0$

Solution Form: $x(t) = A\cos\left(\omega_n t\right) + B\sin\left(\omega_n t\right)$

$$\omega_n = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$$