MEEM 3700
Mechanical Vibrations

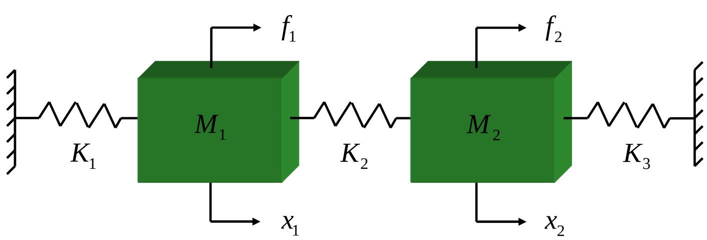
Mohan D. Rao

Chuck Van Karsen

Mechanical Engineering-Engineering Mechanics

Michigan Technological University

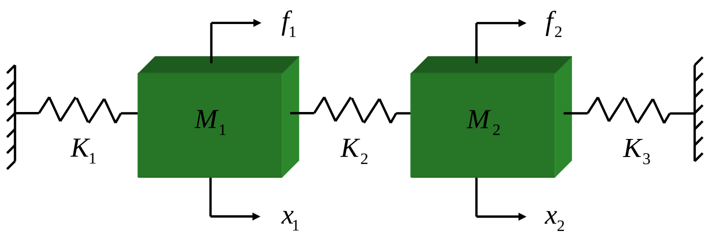
2 DOF Forced Vibration



Given a 2-degree of freedom system with no damping the equations of motion in matrix form can be generalized as:

$$\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

2 DOF Forced Vibration



Assuming harmonic forcing functions of the form:

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \sin(\omega t)$$

The solution (motion of x_1 and x_2) will be of the form:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t)$$

2 DOF Forced Vibration: Solution

Substituting into the equation of motion:

$$\left(-\omega^2 \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t) = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \sin(\omega t)$$

and simplifying

$$\left(-\omega^2 \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

2 DOF Forced Vibration: Impedance Matrix

Define: $$\left[Z(\omega)\right]=\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}-\omega^2\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix}$$

$$\left[Z(\omega)\right]=\begin{bmatrix} k_{11}-\omega^2 m_{11} & k_{12} \\ k_{21} & k_{22}-\omega^2 m_{22} \end{bmatrix}$$

Where $\left[Z(\omega)\right]$ is called the *system* or *impedance* matrix.

Then the response $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$ can be solved for as:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=\left[Z(\omega)\right]^{-1}\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \left[Z(\omega)\right]^{-1}=\frac{adj\left[Z(\omega)\right]}{\det\left[Z(\omega)\right]}$$

Linear Algebra Review: Matrix Operations

Inverse of A → $$[A]^{-1}=\frac{adjoint[A]}{\det[A]}$$

$$\text{adjoint}[A]=\text{transpose of cofactor }[A]$$

$$\text{cofactors}=\pm\text{ minor determinants}$$

$$[A]=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad cof[A]=\begin{pmatrix} +\left(a_{22}a_{33}-a_{23}a_{32}\right) & \cdots \\ -\left(a_{12}a_{33}-a_{13}a_{32}\right) & \ddots \\ \cdots & \cdots \end{pmatrix}$$

MichiganTech

2 DOF Forced Vibration: Impedance Matrix

The inverse of the impedance matrix is written as:

$$\left[Z(\omega)\right]^{-1} = \frac{\begin{bmatrix} k_{22}-\omega^2 m_{22} & -k_{12} \\ -k_{21} & k_{11}-\omega^2 m_{11} \end{bmatrix}}{\left(k_{11}-\omega^2 m_{11}\right)\left(k_{22}-\omega m_{22}\right)-\left(k_{12}k_{21}\right)}$$

Then the response $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$ can be solved for as:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \left[Z(\omega)\right]^{-1} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

MichiganTech

2 DOF Forced Vibration: Impedance Matrix

Therefore each response can be written as:

$$X_1 = \frac{\left(k_{22}-\omega^2 m_{22}\right)F_1 - k_{12}F_2}{\left(k_{11}-\omega^2 m_{11}\right)\left(k_{22}-\omega^2 m_{22}\right)-\left(k_{12}k_{21}\right)}$$

$$X_2 = \frac{\left(k_{11}-\omega^2 m_{11}\right)F_2 - k_{21}F_1}{\left(k_{11}-\omega^2 m_{11}\right)\left(k_{22}-\omega^2 m_{22}\right)-\left(k_{12}k_{21}\right)}$$

2 DOF Forced Vibration: FRF Matrix

Define $\left[H(\omega)\right]=\left[Z(\omega)\right]^{-1}$

as the Frequency Response Function Matrix

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \left[H(\omega)\right]\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Therefore each response can be written as:

$$X_1 = h_{11}F_1 + h_{12}F_2$$

$$X_2 = h_{21}F_1 + h_{22}F_2$$

2 DOF Forced Vibration: FRF

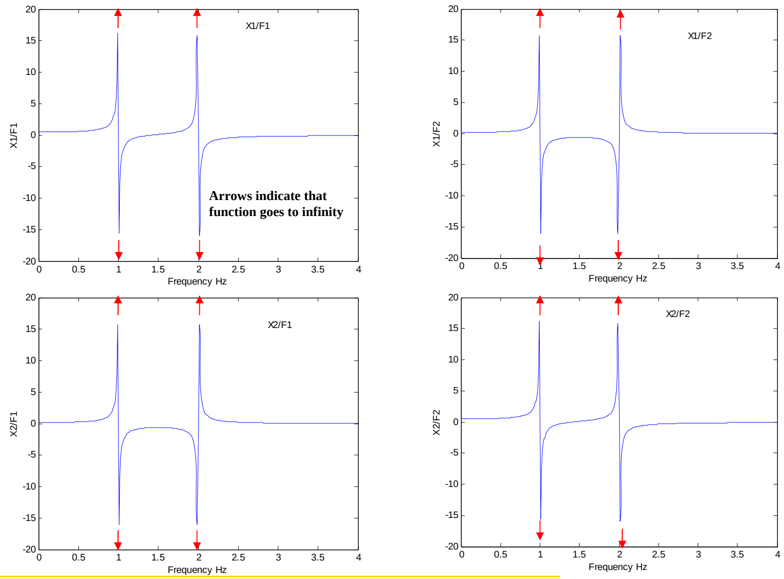
The individual FRF's are defined as:

$$\frac{X_1}{F_1} = h_{11} = \frac{k_{22}-\omega^2 m_{22}}{\left(k_{11}-\omega^2 m_{11}\right)\left(k_{22}-\omega^2 m_{22}\right)-\left(k_{12}k_{21}\right)}$$

$$\frac{X_1}{F_2} = h_{12} = \frac{-k_{12}}{\left(k_{11}-\omega^2 m_{11}\right)\left(k_{22}-\omega^2 m_{22}\right)-\left(k_{12}k_{21}\right)}$$

$$\frac{X_2}{F_1} = h_{21} = \frac{-k_{21}}{\left(k_{11}-\omega^2 m_{11}\right)\left(k_{22}-\omega^2 m_{22}\right)-\left(k_{12}k_{21}\right)}$$

$$\frac{X_2}{F_2} = h_{22} = \frac{k_{11}-\omega^2 m_{11}}{\left(k_{11}-\omega^2 m_{11}\right)\left(k_{22}-\omega^2 m_{22}\right)-\left(k_{12}k_{21}\right)}$$



2 DOF Forced Vibration: Damping

If damping is considered the equations of motion become:

$$\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

And if the forcing functions are harmonic functions (as before) the solution is:

$$\left(-\omega^2 \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} + j\omega \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

2 DOF Forced Vibration: FRF w/ Damping

Define: $$\left[Z(\omega)\right]=\begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}-\omega^2\begin{bmatrix}m_{11} & 0\\ 0 & m_{22}\end{bmatrix}+j\omega\begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{bmatrix}$$

Where $\left[Z(\omega)\right]$ is called the *system* or *impedance* matrix.

Then the response $\begin{Bmatrix}X_1\\ X_2\end{Bmatrix}$ can be solved for as:

$$\begin{Bmatrix}X_1\\ X_2\end{Bmatrix}=\left[Z(\omega)\right]^{-1}\begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

Where $\left[Z(\omega)\right]^{-1}=\left[H(\omega)\right]$ is the Frequency Response Function Matrix

$$\begin{Bmatrix}X_1\\ X_2\end{Bmatrix}=\left[H(\omega)\right]\begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

2 DOF Forced Vibration: FRF w/ Damping

The FRF matrix can be written as:

$$\left[H(\omega)\right]=\frac{adj\left[Z(\omega)\right]}{\det\left[Z(\omega)\right]}$$

This is expanded to:

$$\left[H(\omega)\right]=\frac{\begin{bmatrix}k_{22}-\omega^2 m_{22}+j\omega c_{22} & -\left(j\omega c_{12}+k_{12}\right)\\ -\left(j\omega c_{21}+k_{21}\right) & k_{11}-\omega^2 m_{11}+j\omega c_{11}\end{bmatrix}}{\left(k_{11}-\omega^2 m_{11}+j\omega c_{11}\right)\left(k_{22}-\omega m_{22}+j\omega c_{22}\right)-\left(j\omega c_{12}+k_{12}\right)\left(j\omega c_{21}+k_{21}\right)}$$

Therefore each response can be written as:

$$X_1=h_{11}F_1+h_{12}F_2$$

$$X_2=h_{21}F_1+h_{22}F_2$$

2 DOF Forced Vibration: FRF w/ Damping

The individual FRF's are defined as:

$$\frac{X_1}{F_1} = h_{11} = \frac{(k_{22} - \omega^2 m_{22}) + j\omega c_{22}}{(k_{11} - \omega^2 m_{11})(k_{22} - \omega^2 m_{22}) - (j\omega c_{12} + k_{12})(j\omega c_{21} + k_{21})}$$

$$\frac{X_1}{F_2} = h_{12} = \frac{-(k_{12} + j\omega c_{12})}{(k_{11} - \omega^2 m_{11})(k_{22} - \omega^2 m_{22}) - (j\omega c_{12} + k_{12})(j\omega c_{21} + k_{21})}$$

$$\frac{X_2}{F_1} = h_{21} = \frac{-(k_{21} + j\omega c_{21})}{(k_{11} - \omega^2 m_{11})(k_{22} - \omega^2 m_{22}) - (j\omega c_{12} + k_{12})(j\omega c_{21} + k_{21})}$$

$$\frac{X_2}{F_2} = h_{22} = \frac{(k_{11} - \omega^2 m_{11}) + j\omega c_{11}}{(k_{11} - \omega^2 m_{11})(k_{22} - \omega^2 m_{22}) - (j\omega c_{12} + k_{12})(j\omega c_{21} + k_{21})}$$