MEEM 3700
Mechanical Vibrations

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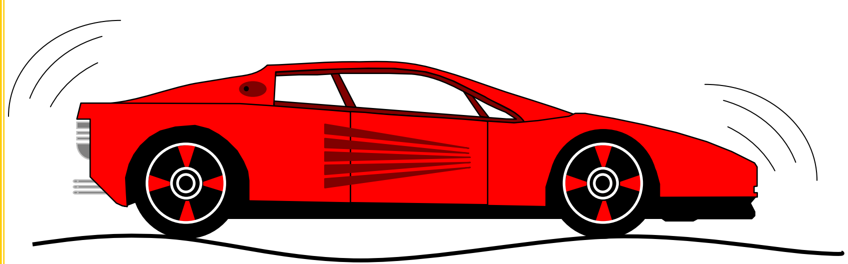
Multiple Degree of Freedom Systems



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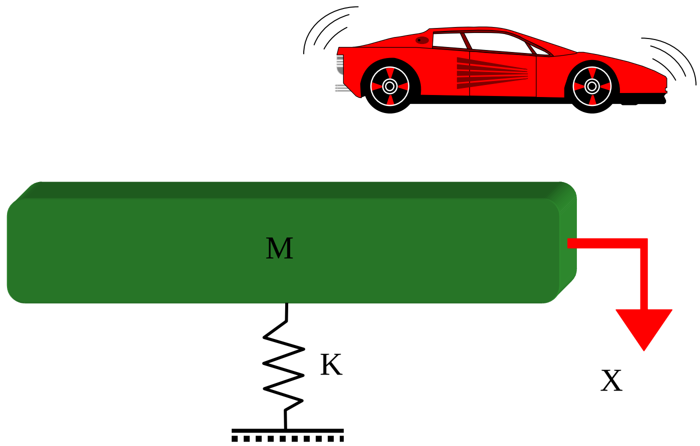
Vibrating Systems: Single Degree of Freedom (SDOF)



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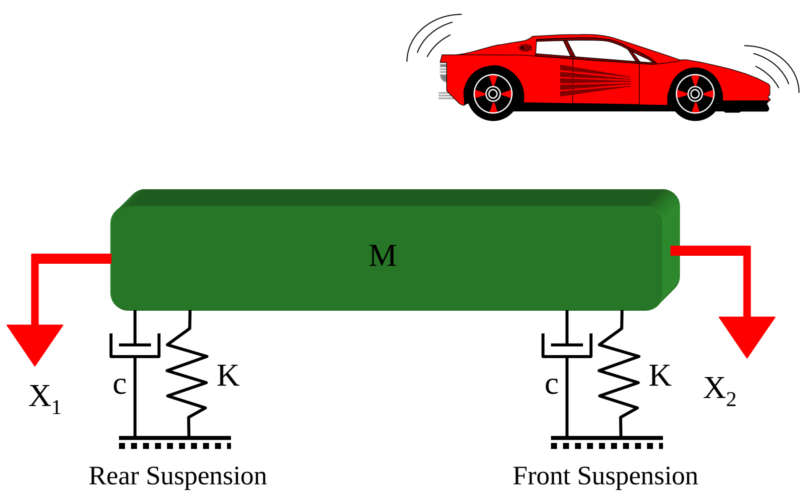
Vibrating Systems: Multi Degree of Freedom (MDOF)



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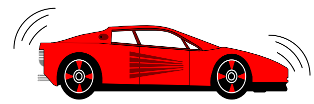
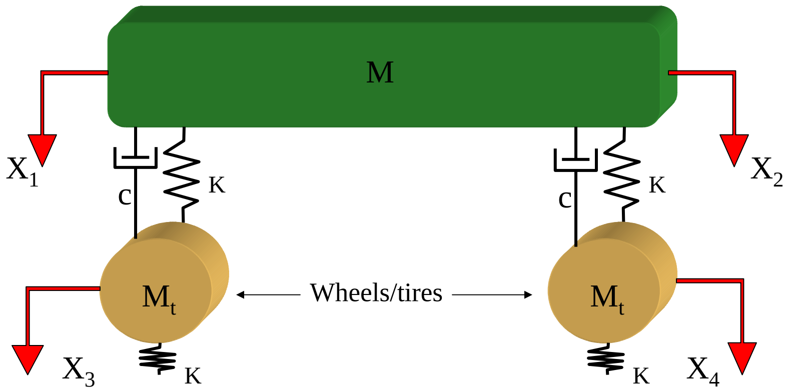
Vibrating Systems: Multi Degree of Freedom (MDOF)





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Lecture 17- Multiple DOF Free Vibration

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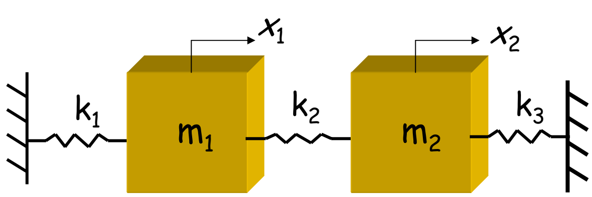
2 Degrees of Freedom: Free Vibration



Given a set of initial conditions, determine the free vibration response of the system.

$$x_1(0) \quad \dot{x}_1(0) \quad x_2(0) \quad \dot{x}_2(0)$$

$$x_1(t) = ? \qquad x_2(t) = ?$$

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Lecture 17- Multiple DOF Free Vibration

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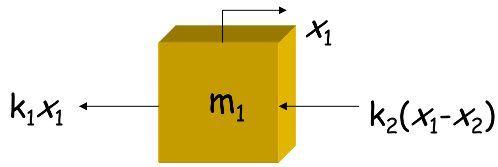
2 Degrees of Freedom: Free Vibration

F.B.D. for mass 1



$$\sum F_{x_1} = \sum F_{eff} = m_1\ddot{x}_1$$

$$-k_1 x - k_2\left(x_1 - x_2\right) = m_1\ddot{x}_1$$

$$m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2 x_2 = 0$$ EOM #1

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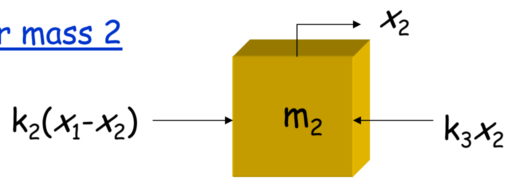
2 Degrees of Freedom: Free Vibration

F.B.D. for mass 2



$$\sum F_{x_2} = \sum F_{eff}$$

$$k_2\left(x_1 - x_2\right) - k_3 x_2 = m_2\ddot{x}_2$$

$$m_2\ddot{x}_2 + \left(k_2 + k_3\right)x_2 - k_2 x_1 = 0$$ EOM #2

2 Degrees of Freedom: Free Vibration

$$m_1\ddot{x}_1 + (k_2 + k_1)x_1 - k_2 x_2 = 0$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = 0$$

2^{nd} order differential equations

Linear

Homogeneous

Constant coefficients

Coupled

2 Degrees of Freedom: Free Vibration

For a solution to simultaneous equations, rewrite in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[m]\{\ddot{x}\} + [k]\{x\} = 0$$

Mass matrix

stiffness matrix

Acceleration vector

Displacement vector

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2 Degrees of Freedom: Free Vibration

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

assume solution and substitute

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi)$$

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi)$$

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2 Degrees of Freedom: Free Vibration

$$-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi) + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 + k_3 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

A solution to this set of equations exists if the determinant of the coefficient matrix is equal to zero.

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2 Degrees of Freedom: Free Vibration

$$\det\begin{bmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2+k_3-\omega^2 m_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(k_1+k_2-\omega^2 m_1\right)\left(k_2+k_3-\omega^2 m_2\right)-k_2^2=0$$

$$let: \lambda=\omega^2$$

$$\left(k_1+k_2-\lambda m_1\right)\left(k_2+k_3-\lambda m_2\right)-k_2^2=0$$

$$\left(k_1+k_2\right)\left(k_2+k_3\right)-\lambda m_2\left(k_1+k_2\right)-\lambda^2 m_1 m_2-k_2^2=0$$

$$m_1 m_2\lambda^2-\lambda\left[m_2\left(k_1+k_2\right)+m_1\left(k_2+k_3\right)\right]+\left(k_1+k_2\right)\left(k_2+k_3\right)-k_2^2=0$$

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2 Degrees of Freedom: Free Vibration

$$m_1 m_2\lambda^2-\lambda\left[m_2\left(k_1+k_2\right)+m_1\left(k_2+k_3\right)\right]+\left(k_1+k_2\right)\left(k_2+k_3\right)-k_2^2=0$$

A quadratic equation in λ

$$\lambda_{1,2}=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\lambda_{1,2}=\frac{-\left[m_2\left(k_1+k_2\right)+m_1\left(k_2+k_3\right)\right]}{2m_1m_2}\pm\frac{\sqrt{\left[m_2\left(k_1+k_2\right)+m_1\left(k_2+k_3\right)\right]^2-4m_1m_2\left(\left(k_1+k_2\right)\left(k_2+k_3\right)-k_2^2\right)}}{2m_1m_2}$$

Recall → $\lambda=\omega^2$

Therefore → $\omega_1=\sqrt{\lambda_1}\quad \omega_2=\sqrt{\lambda_2}$

ω_1 and ω_2 are the natural frequencies of the system

2 Degrees of Freedom: Free Vibration

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi)$$

for each value of ω_1 and ω_2 there is a corresponding vector $\begin{Bmatrix} X_1^1 \\ X_2^1 \end{Bmatrix}$ $\begin{Bmatrix} X_1^2 \\ X_2^2 \end{Bmatrix}$

$\begin{Bmatrix} X_1^1 \\ X_2^1 \end{Bmatrix}$ $\begin{Bmatrix} X_1^2 \\ X_2^2 \end{Bmatrix}$ are called the mode shapes of the system

The total solution is:

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = A \begin{Bmatrix} X_1^1 \\ X_2^1 \end{Bmatrix} \sin(\omega_1 t + \phi_1) + B \begin{Bmatrix} X_1^2 \\ X_2^2 \end{Bmatrix} \sin(\omega_2 t + \phi_2)$$

2 Degrees of Freedom: Free Vibration

Alternative Solution method: Eigen-Solution

$$\left(-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ x_2 \end{Bmatrix} = \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ x_2 \end{Bmatrix}$$

Let $\lambda = \omega^2$

$$[k]\{x\} = \lambda [m]\{x\}$$

Eigenvalue problem

2 Degrees of Freedom: Free Vibration

$$[k]\{x\} = \lambda[m]\{x\}$$

eigenvalues = $\lambda_1, \lambda_2, \lambda_3, ...\lambda_N$ where N = the number of degrees of freedom

and associated eigenvectors $\begin{Bmatrix} X_1^1 \\ X_2^1 \end{Bmatrix}, \begin{Bmatrix} X_1^2 \\ X_2^2 \end{Bmatrix}, \begin{Bmatrix} X_1^3 \\ X_2^3 \end{Bmatrix}, ... \begin{Bmatrix} X_1^N \\ X_2^N \end{Bmatrix}$

The natural frequencies are the square root of the eigenvalues

$$\omega_i = \sqrt{\lambda_i}$$

The eigenvectors are also called the mode shapes of the system