MEEM 3700
Mechanical Vibrations

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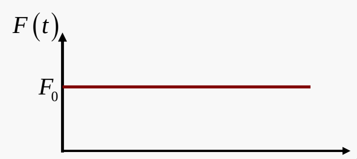
Laplace Transform: Example 1

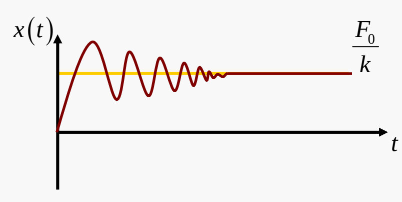
$$m\ddot{x}+c\dot{x}+kx=F_0$$

$$x(0)=0$$

$$\dot{x}(0)=0$$



Step Function



Laplace Transform: Example 1

$$m\ddot{x}+c\dot{x}+kx=F_0$$

$$m\left(s^2X(s)-sx(0)-\dot{x}(0)\right)+c\left(sX(s)-x(0)\right)+kX(s)=\frac{F_0}{s}$$

$$\left(ms^2+cs+k\right)X(s)=\frac{F_0}{s}$$

$$X(s)=\frac{F_0}{s\left(ms^2+cs+k\right)}$$

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Laplace Transform: Example 1

$$X(s)=\frac{F_0}{s\left(ms^2+cs+k\right)}=\frac{F_0/m}{s\left(s^2+\frac{c}{m}s+\frac{k}{m}\right)}$$

$$\lambda=-\frac{c}{2m}\pm\frac{\sqrt{\left(c/m\right)^2-4\left(k/m\right)}}{2}$$

$$\lambda=-\zeta\omega_n\pm j\omega_d$$

If $\zeta<1$

$$X(s)=\frac{F_0/m}{s\left(s-\lambda\right)\left(s-\lambda^*\right)}$$

Partial Fraction Expansion

$$X(s)=\frac{F_0/m}{s\left(s-\lambda\right)\left(s-\lambda^*\right)}=\frac{A}{s}+\frac{B}{s-\lambda}+\frac{C}{s-\lambda^*}$$

$$x(t)=A+Be^{\lambda t}+Ce^{\lambda^* t}$$

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Laplace Transform: Example 1

Pole – Residue Theorem A, B, C ⟹ **Residues**

$$\frac{F_0/m}{s(s-\lambda)(s-\lambda^*)} = \frac{A}{s} + \frac{B}{s-\lambda} + \frac{C}{s-\lambda^*}$$

multiply by s

evaluate @ $s = 0$ **(root)**

$$\frac{F_0/m}{(-\lambda)(-\lambda^*)} = \frac{As}{s} + \frac{Bs}{s-\lambda} + \frac{Cs}{s-\lambda^*} \rightarrow A = \frac{F_0/m}{\lambda\lambda^*}$$

$$\left.\frac{F_0/m\,(s-\lambda)}{s(s-\lambda)(s-\lambda^*)}\right|_{s=\lambda} = \left.\frac{A(s-\lambda)}{s} + B + \frac{C(s-\lambda)}{s-\lambda^*}\right|_{s=\lambda}$$

Laplace Transform: Example 1

$$\frac{F_0/m}{\lambda(\lambda-\lambda^*)} = B \qquad A = \frac{F_0/m}{\lambda\lambda^*}$$

$$\frac{F_0/m}{\lambda^*(\lambda^*-\lambda)} = C \qquad A = \frac{F_0}{k}$$

$$x(t) = \frac{F_0}{k} + Be^{\lambda t} + Ce^{\lambda^* t}$$

$$x(t) = \frac{F_0}{k} + B\left(e^{-\zeta\omega_n t}e^{j\omega_d t}\right) + C\left(e^{-\zeta\omega_n t}e^{j\omega_d t}\right)$$

Use $e^{\pm j\omega t} = \cos(\omega t) \pm j\sin(\omega t)$

Laplace Transform: Example 1

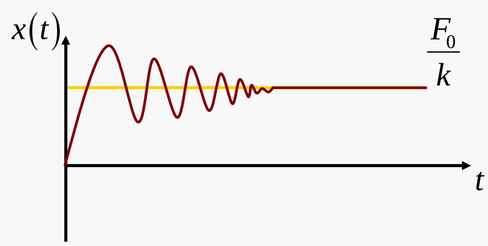
$$x(t)=\frac{F_0}{k}+e^{-\zeta\omega_n t}\left[Be^{j\omega_d t}+Ce^{-j\omega_d t}\right]$$

$$x(t)=\frac{F_0}{k}+e^{-\zeta\omega_n t}\left[(B+C)\cos(\omega_d t)+j(B-C)\sin(\omega_d t)\right]$$

Substitute for B & C, rearrange & plot

$$\frac{F_0/m}{\lambda(\lambda-\lambda^*)}=B$$

$$\frac{F_0/m}{\lambda^*(\lambda^*-\lambda)}=C$$



Laplace Transform: Example 2

SDOF system subjected to a harmonic excitation

$$10\ddot{x}+\dot{x}+2000x=50\cos(20t) \qquad x(0)=2\ ,\ \dot{x}(0)=0$$

$$x(t)=?$$

$$m\left(s^2X(s)-2s+0\right)+c\left(sX(s)-2\right)+kX(s)=\frac{50s}{s^2+20^2}$$

$$X(s)=\frac{50s}{\left(10s^2+s+2000\right)\left(s^2+400\right)}+\frac{20s+2}{\left(10s^2+s+2000\right)}$$

$$X(s)=\frac{5s}{(s-\lambda)(s-\lambda^*)(s-j\omega)(s+j\omega)}+\frac{2s+0.2}{(s-\lambda)(s-\lambda^*)}$$

$$\lambda=-0.05+14.14j \qquad \omega=20$$

Laplace Transform: Example 2

$$X(s)=\frac{A}{s-\lambda}+\frac{B}{s-\lambda^*}+\frac{C}{s-j\omega}+\frac{D}{s+j\omega}+\frac{E}{s-\lambda}+\frac{F}{s-\lambda^*}$$

$$B=A^* \quad , \quad D=C^* \quad , \quad F=E^*$$

$$A=\frac{5\lambda}{(\lambda-\lambda^*)(\lambda-j20)(\lambda+j20)}=0.0125+0.0001j$$

$$C=\frac{5(j20)}{(20j-\lambda)(20j-\lambda^*)(20j+20j)}=-0.0125-0.0001j$$

$$E=\frac{2\lambda+0.2}{(\lambda-\lambda^*)}=1.0-0.0035j$$

$B=0.0125-0.0001j \qquad D=-0.0125+0.0001j \qquad F=1.0+0.0035j$

Laplace Transform: Example 2

$$X(s)=\frac{A+E}{(s-\lambda)}+\frac{B+F}{(s-\lambda^*)}+\frac{C}{s-j\omega}+\frac{D}{s+j\omega}$$

$$x(t)=(A+E)e^{\lambda t}+(B+F)e^{\lambda^* t}+Ce^{j\omega t}+De^{-j\omega t}$$

$$x(t)=e^{-0.5t}\left[(A+E)e^{14.14jt}+(B+F)e^{-j14.14t}\right]+Ce^{20jt}+De^{-20jt}$$

$$e^{\pm j\omega t}=\cos(\omega t)\pm j\sin(\omega t)$$

Steady state part

Transient part

$$x(t)=e^{-0.5t}\left(2.025\cos(14.14t)+0.007\sin(14.14t)\right)+0.025\cos(20t-\pi)$$

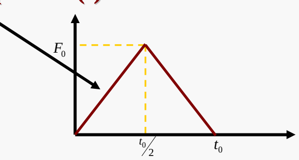
This is similar to what was obtained before by the classical method
$x(t) = x_{transient} + x_{steady\ state}$

Laplace Transform: Example 3– Triangular Pulse Excitaiton

Determine the response of an undamped mass-spring system at rest in equilibrium to a symmetric triangular pulse F(t) of total duration t_0

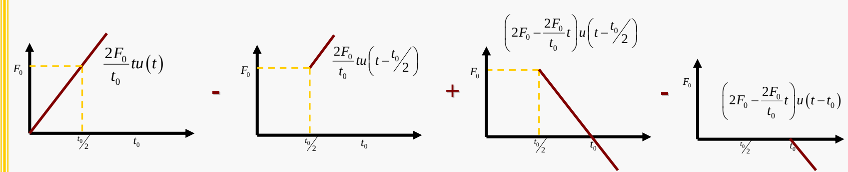
Given: $m\ddot{x} + kx = F(t)$ and , $x(0) = 0$, $\dot{x}(0) = 0$

Find: $x(t)$



Solution: The graphical breakdown of the triangular pulse using unit step functions is shown below.

$u(t) =$



Laplace Transform: Example 3

The mathematical form of the expression is

$$F(t) = \frac{2F_0 t}{t_0}\left[u(t) - u\left(t - \frac{t_0}{2}\right)\right] + \left(\frac{-2F_0 t}{t_0} + 2F_0\right)\left[u\left(t - \frac{t_0}{2}\right) - u(t - t_0)\right]$$

$$= \frac{2F_0 t}{t_0} u(t) - \frac{4F_0}{t_0}\left(t - \frac{t_0}{2}\right) u\left(t - \frac{t_0}{2}\right) + \frac{2F_0}{t_0}(t - t_0) u(t - t_0)$$

The Laplace transform of F(t) is obtained using the second shifting theorem and the proper transform from the Laplace transform table.

$$F(s) = \frac{2F_0}{t_0 s^2}\left[1 - 2e^{-s\frac{t_0}{2}} + e^{-st_0}\right]$$

The transform of the system response is

$$X(s) = \frac{F(s)}{m\left(s^2 + \omega_n^2\right)}$$

Second Shifting Theorem
If L[f(t)]= F(s) and g(t)= f(t-a) for a>0 =o for t<0, then L[g(t)]= e^{-as} F(S)

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Laplace Transform: Example 3

Use partial fraction decomposition to express

$$\frac{1}{s^2\left(s^2+\omega_n^2\right)}=\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+\omega_n{}^2}$$

$$solving, A=-C=0,$$

$$B=1/\omega_n{}^2, D=-B$$

Hence $$\frac{1}{s^2\left(s^2+\omega_n^2\right)}=\frac{1}{\omega_n^2}\left(\frac{1}{s^2}-\frac{1}{\left(s^2+\omega_n^2\right)}\right)$$

$$X\left(s\right)=\frac{2F_0}{mt_0\omega_n^2}\left(\frac{1}{s^2}-\frac{1}{s^2+\omega_n^2}\right)^{\left(1-2e^{-s\frac{t_0}{2}}+e^{-st_0}\right)}$$

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Laplace Transform: Example 3

Application of the first shifting theorem and transform pairs from the Laplace transform table are used to invert the transform and obtain the system response:

$$x\left(t\right)=\frac{2F_0}{mt_0\omega_n^2}\left\{\left(t-\frac{1}{\omega_n}\sin\left(\omega_n t\right)\right)u\left(t\right)-2\left[t-\frac{t_0}{2}-\frac{1}{\omega_n}\sin\left(\omega_n\left(t-\frac{t_0}{2}\right)\right)\right]u\left(t-\frac{t_0}{2}\right)+\left[t-t_0-\frac{1}{\omega_n}\sin\left(\omega_n\left(t-t_0\right)\right)\right]u\left(t-t_0\right)\right\}$$

This example illustrates application of the Laplace transform to determine the response of an undamped one-degree-of-freedom system subject to an excitation whose form changes with time.

First Shifting Theorem
If L[x(t)]= X(s), then
L[e^{at} f(t)]= F(s-a)

Partial Fraction Expansion with Complex Roots

$$X(s) = \frac{1}{\left(s^2 + a_1 s + a_2\right)}$$

Here, the polynomial has two roots that are complex.

Solution Procedure

Write: $s^2 + a_1 s + a_2 = (s + a_1/2)^2 + (a_2 - a_1^4/4)$

$= (s+a)^2 + b^2 \rightarrow a = a_1/2$ and $b = (a_2 - a_1^2/4)$

Then Write

$$\frac{1}{\left(s^2 + a_1 s + a_2\right)} = \frac{A(s+a)}{(s+a)^2 + b^2} + \frac{Bb}{(s+a)^2 + b^2}$$

Obtain A and B the usual way

The Inverse Laplace Transform is given by

$x(t) = L[X(s)] = A\, e^{-at} \cos bt + B\, e^{-at} \sin bt$