MEEM 3700
Mechanical Vibrations

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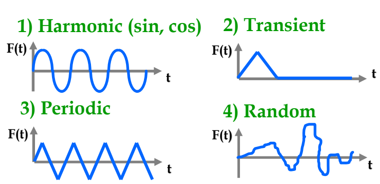
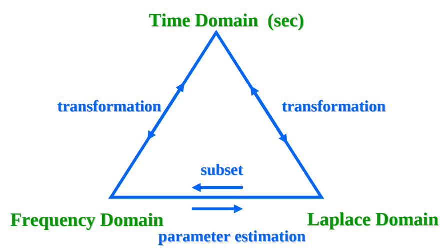
Laplace Transform





Each domain can be thought of as a different coordinate system that is used to view or describe the characteristics of a system or event.

The Time Domain is usually the basis for a description of a systems dynamic behavior, e.g. differential equation of motion. Events are measured as a function of time.

The Frequency Domain highlights the periodic characteristics of the system or event.

The Laplace Domain describes the system in terms of frequency and damping information (poles) or Mode Shape information (residues).

Laplace Transform

The Laplace transform of a function *x(t)* is defined as:

$$X(s) = L\left[x(t)\right] = \int_0^{\infty} x(t)\,\underbrace{e^{-st}}_{\text{Kernel of transformation}}\,dt$$

Kernel of transformation

Where *s* (a complex function) is a subsidiary variable.

$$s = -\sigma + j\omega$$

Units are rad/sec

Laplace Transform: Example

To solve: $m\ddot{x} + c\dot{x} + kx = F(t)$ ⟹

$$L\left[\dot{x}(t)\right] = \int_0^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$

$$= e^{-st}x(t)\Big|_0^{\infty} - \int_0^{\infty} -se^{-st}x(t)\,dt$$

$$= e^{-st}x(t)\Big|_0^{\infty} + s\int_0^{\infty} e^{-st}x(t)\,dt$$

$$= sX(s) - x(0)$$

Similarly $L\left[\ddot{x}(t)\right] = s^2X(s) - sx(0) - \dot{x}(0)$

Laplace Transform: Example

In general: $\frac{d^m x(t)}{dx^m} = s^m X(s) - s^{m-1}x(0) - s^{m-2}\dot{x}(0) - \ldots - x^{m-1}(0)$

The EOM becomes: $mL\left[\ddot{x}(t)\right] + cL\left[\dot{x}(t)\right] + kL\left[x(t)\right] = L\left[F(t)\right]$

i.e. $\left(ms^2 + cs + k\right)X(s) = F(s) + m\ddot{x}(0) + (ms + c)x(0)$

For the steady-state solution: $\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} = \frac{1}{m\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$

(assuming $x(0) = \dot{x}(0) = 0$)

$$X(s) = H(s)F(s) \quad \text{where} \quad H(s) = \frac{1}{ms^2 + cs + k}$$

$$x(t) = L^{-1}\left[X(s)\right] = L^{-1}\left[\frac{F(s)}{ms^2 + cs + k}\right]$$

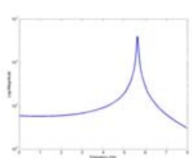
$H(s) = \frac{1}{ms^2 + cs + k}$ is called the Transfer Function

if $s = j\omega$

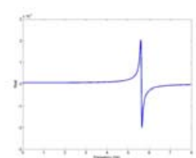
Recall from Lecture 12

$H(j\omega) = \frac{1}{-\omega^2 m + jc\omega + k}$ is called the Frequency Response Function

$|H(j\omega)|$



Real $(H(j\omega))$



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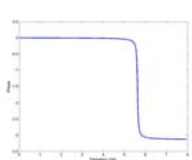
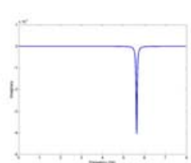


Imag $(H(j\omega))$



Laplace Transform: Example

Special Case: Undamped System, Free Vibration

$$m\ddot{x}+kx=0 \quad \Rightarrow \quad \ddot{x}+\omega_n^2=0 \quad \textbf{or} \quad s^2X(s)-sx(0)-\dot{x}(0)+\omega_n^2X(s)=0$$

$$X(s)=\frac{x(0)+s\dot{x}(0)}{s^2+\omega_n^2} \qquad x(t)=L^{-1}\left[x(s)\right]$$

Use the Table of Laplace Transforms to get *x(t)* from *X(s)*

i.e. $$X(s)=\frac{\dot{x}(0)+sx(0)}{s^2+\omega_n^2}$$

$$x(t)=x(0)\cos(\omega_n t)+\frac{\dot{x}(0)}{\omega_n}\sin(\omega_n t)$$

From the table:

$$L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos(at)$$

$$L^{-1}\left(\frac{a}{s^2+a^2}\right)=\sin(at)$$

Laplace Transform: Method of Partial Fractions

e.g. If $X(s)=\frac{s+1}{s(s+2)}$ **find *x(t)***

This does not appear in the Laplace Transform table.

To solve, we use Partial Fractions

$$\frac{s+1}{s(s+2)}=\frac{A}{s}+\frac{B}{(s+2)} \quad \Rightarrow \quad \frac{s+1}{s(s+2)}=\frac{A(s+2)+sB}{s(s+2)}$$

i.e. $s+1=A(s+2)+sB=(A+B)s+2A$

Equating Coefficients: $A+B=1 \quad 2A=1 \quad \text{A=B=}\frac{1}{2}$

Hence $X(s)=\frac{1}{2s}+\frac{1}{2(s+2)}$ **Now, from table:** $x(t)=\frac{1}{2}\left(1+e^{-2t}\right)$

Laplace Transform: Other Rules – Multiple Roots

e.g. $\dfrac{\omega^2}{s(s+\omega)^2} \Rightarrow \dfrac{\omega^2}{s(s+\omega)^2} = \dfrac{A}{s} + \dfrac{B}{(s+\omega)} + \dfrac{C}{(s+\omega)^2}$

Multiply both sides by $s(s+\omega)^2$

$$\omega^2 = A(s+\omega)^2 + Bs(s+\omega) + Cs$$

$$= A(s^2 + 2s\omega + \omega^2) + Bs^2 + Bs\omega + Cs$$

$$= s^2(A+B) + s(2A\omega + B\omega + C) + A\omega^2$$

Equating Coefficients: $A=1 \qquad A+B=0 \qquad A=-B=-1$

$$2A\omega + B\omega + C = 0 \Rightarrow C = -(2A\omega + B\omega)$$

$$= -\omega(2A+B)$$

$$= -\omega$$

$$\boxed{A=1 \qquad B=-1 \qquad C=-\omega}$$

Laplace Transform: Other Rules – Multiple Roots (continued)

From the equation coefficients: $A=1 \qquad B=-1 \qquad C=-\omega$

Hence,

$$X(s) = \frac{\omega^2}{s(s+\omega)^2} = \frac{1}{s} - \frac{1}{(s+\omega)} - \frac{\omega}{(s+\omega)^2}$$

From table,

$$x(t) = 1 - e^{-\omega t} - \omega t e^{-\omega t}$$

Laplace Transform: Another example involving Partial Fractions

$$X(s)=\frac{s+3}{(s+1)(s^2+2s+5)} \Rightarrow \frac{s+3}{(s+1)(s^2+2s+5)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+2s+5}$$

$$(s+3)=(A+B)s^2+(2A+B+C)s+(5A+C)$$

$$A+B=0 \qquad 2A+B+C=1 \qquad 5A+C=0$$

$$A=\frac{1}{2} \qquad B=-\frac{1}{2} \qquad C=\frac{1}{2}$$

$$X(s)=\frac{1}{2(s+1)}+\frac{1-s}{2(s^2+2s+5)}=\frac{1}{2(s+1)}-\frac{s+1-2}{2(s^2+2s+5)}$$

$$=\frac{1}{2(s+1)}-\left(\frac{1}{2}\right)\frac{s+1}{(s+1)^2+4}+\frac{2}{2(s^2+2s+5)}$$

Invert this using the table to get

$$x(t)=\frac{1}{2}e^{-t}-\frac{1}{2}e^{-t}\cos(2t)+e^{-t}\sin(2t)$$

Laplace Transform: Other Rules

If $F(s)=\dfrac{P(s)}{Q(s)}$

express $Q(s)=(s-a_1)(s-a_2)\ldots(s-a_n)$

where a_1, a_2, a_3, … a_n are distinct roots of $Q(s)$

Then $f(t)=\sum_{k=1}^{n}\dfrac{P(a_k)}{Q'(a_k)}e^{a_k t}$ $\qquad Q'=\dfrac{dQ}{ds}$

e.g. $\dfrac{P(s)}{Q(s)}=\dfrac{F_0}{ms^2+cs+k}$ $\qquad a_{1,2}=-\dfrac{c}{2m}\pm\sqrt{\dfrac{c^2-4km}{2m}}$

$$x(t)=\frac{F_0 e^{a_1 t}}{2ma_1+c}+\frac{F_0 e^{a_2 t}}{2ma_2+c}$$

Solution of an O.D.E. using Laplace Transforms

For example: $\ddot{x}+4\dot{x}+3x=0 \qquad x(0)=0 \qquad \dot{x}(0)=1$

Laplace Transform Method

$$\left[s^2X(s)-sx(0)-\dot{x}(0)\right]+4\left[sX(s)-x(0)\right]+3X(s)=0$$

$$s^2X(s)-1+4sX(s)-0+3X(s)=0$$

$$X(s)\left[s^2+4s+3\right]=1$$

$$X(s)=\frac{1}{s^2+4s+3}$$

$$x(t)=L^{-1}\left[\frac{1}{s^2+4s+3}\right]$$

expand $\dfrac{1}{s^2+4s+3}$ **as partial fractions.**

Traditional Method

$m=1 \qquad \omega_n=\sqrt{3} \qquad 2\zeta\omega_n=4$

Overdamped System $\zeta=\dfrac{4}{2\sqrt{2}}\geq 1$

$\alpha^2+4\alpha+3=0$

$\alpha_{1,2}=-1,\ -3$

$$x(t)=Ae^{-t}+Be^{-3t}$$

$$x(0)=0 \qquad \dot{x}(0)=1$$

$$A+B=0 \qquad A=-B$$

$$\dot{x}(0)=1=-A-3B$$

$$A=\frac{1}{2} \qquad B=-\frac{1}{2}$$

$$x(t)=\frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t}$$

Solution of an O.D.E. using Laplace Transforms

Using partial fractions: $X(s)=\dfrac{1}{s^2+4s+3}=\dfrac{A}{(s+1)}+\dfrac{B}{(s+3)}$

$A(s+3)+B(s+1)=1$ ***s = -1*** **and** ***s = -3*** **are the roots**

put $s=-1 \quad \Rightarrow \quad 2A=1 \qquad A=\dfrac{1}{2}$

put $s=-3 \quad \Rightarrow \quad -2B=1 \qquad B=-\dfrac{1}{2}$

$$x(t)=L^{-1}\left[\frac{1/2}{(s+1)}-\frac{1/2}{(s+3)}\right]$$

from table ⇒ $x(t)=\dfrac{1}{2}e^{-t}-\dfrac{1}{2}e^{-3t}$ **Same as before**

Solution of an O.D.E. using Laplace Transforms

Alternate Method:

If $X(s) = \dfrac{P(s)}{Q(s)}$

$$x(t) = \sum_{k=1}^{n} \frac{P(a_k)}{Q'(a_k)} e^{a_k t}$$

$$Q' = \frac{dQ}{ds}$$

Here $a_1 = -1 \quad P(s) = 1$

$a_2 = -3 \quad Q(s) = s^2 + 4s + 3$

$$x(t) = \frac{1}{2s+4\big|_{s=-1}} e^{-t} + \frac{1}{2s+4\big|_{s=-3}} e^{-3t}$$

$$= \frac{e^{-t}}{2} - \frac{1}{2} e^{-3t}$$

Solution of an O.D.E. using Laplace Transforms: Example 2

$$\ddot{x} + 4\dot{x} + 3x = e^{-t} \xrightarrow{LT} \left[s^2 X(s) - sx(0) - \dot{x}(0)\right] + 4\left[sX(s) - x(0)\right] + 3X(s) = L\left(e^{-t}\right)$$

with

$x(0) = \dot{x}(0) = 1$

Here

$\omega_n = \sqrt{3}$

and it is an

Overdamped System

$\zeta = \dfrac{4}{2\sqrt{3}} > 1$

$$\left(s^2 X(s) - s - 1\right) + 4\left(sX(s) - 1\right) + 3(Xs) = \frac{1}{s+1}$$

from table

$$X(s)\left[s^2 + 4s + 3\right] - s - 5 = \frac{1}{s+1}$$

$$X(s)\left[s^2 + 4s + 3\right] = \frac{1}{(s+1)} + (s+5) = \frac{s^2 + 6s + 6}{(s+1)}$$

$$X(s) = \frac{s^2 + 6s + 6}{(s+1)(s^2 + 4s + 3)} = \frac{s^2 + 6s + 6}{(s+1)^2 (s+3)}$$

$$x(t) = L^{-1}\left[X(s)\right]$$

Solution of an O.D.E. using Laplace Transforms: Example 2

Using Partial Fractions: $$\frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+3)}$$

$$A(s+1)(s+3) + B(s+3) + C(s+1)^2 = s^2+6s+6$$

put $s=-1 \quad \Rightarrow \quad 2B = 1 \qquad B = \frac{1}{2}$

put $s=-3 \quad \Rightarrow \quad 4C = 9-18+6 \qquad C = -\frac{3}{4}$

Equation Coefficients of s^2: $A + C = 1 \qquad A = 1 - C = 1 + \tfrac{3}{4} = \tfrac{7}{4}$

$$x(t) = L^{-1}\left[\frac{7/4}{(s+1)} + \frac{1/2}{(s+1)^2} + \frac{-3/4}{(s+3)}\right]$$

From table,

$$x(t) = \frac{7}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{3}{4}e^{-3t}$$

time delay theorem ⇒ $L^{-1}F(s-a) = e^{at}L^{-1}F(s)$

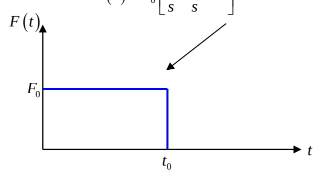
Laplace Transform Method: Example 3

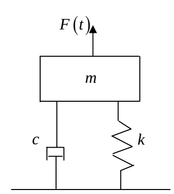
Determine the response of the harmonic oscillator (*m*, *c*, *k*) to the excitation shown in the figure below, a constant force F_0 acting for time t_0.

Solution **The differential equation of motion is** $m\ddot{x} + c\dot{x} + kx = F(t)$ **. We take the Laplace Transform of both sides of the equation, observing that** $L[F(t)] = \frac{F_0(1-e^{-t_0 s})}{s}$.

$f(t) = F_0[u(t) - u(t-t_0)]$

$F(s) = F_0\left[\frac{1}{s} - \frac{1}{s}e^{-t_0 s}\right] \Rightarrow$ Time Delay





Therefore $$m[-sx(0) - \dot{x}(0) + s^2X(s)] + c[-x(0) + sX(s)] + kX(s) = \frac{F_0(1-e^{-t_0 s})}{s}$$

or $$(ms^2 + cs + k)X(s) = \frac{F_0(1-e^{-t_0 s})}{s} + (ms+c)x(0) + m\dot{x}(0)$$

Laplace Transform Method: Example 3

But $x(0)=0$ and $\dot{x}(0)=0$; **therefore,**

$$X(s)=\frac{\left(1-e^{-t_0 s}\right)F_0}{s\left(ms^2+cs+k\right)}=X_1(s)+X_2(s)$$ **, where**

$$X_1(s)=\frac{F_0}{s\left(ms^2+cs+k\right)} \quad , \quad X_2(s)=\frac{-F_0e^{-t_0 s}}{s\left(ms^2+cs+k\right)}$$

To invert these, use the standard form.

Laplace Transform Method: Example 3

If $F(s)=\dfrac{P(s)}{Q(s)}$, then $F(t)=\displaystyle\sum_{k=1}^{n}\frac{P(a_k)}{\dot{Q}(a_k)}e^{a_k}$, For $P(s)=F_0$, $Q(s)=ms^3+cs^2+ks$

$$Q=3ms^2+2cs+k$$

$$a_1=0, \qquad a_{2,3}=\frac{-c}{2m}\pm\frac{\left(c^2-4km\right)^{1/2}}{2m}$$

***a_1, a_2, … a_n* are distinct roots of *Q(s)*.**

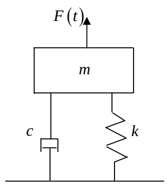
$$x_1(t)=F_0\left(\frac{1}{k}+\frac{e^{a_2 t}}{3ma_2^2+2ca_2+k}+\frac{e^{a_3 t}}{3ma_3^2+2ca_3+k}\right)$$

For *$X_2(s)$*, we note that it is of the form, $X_2(s)=e^{-at}X_1(s)$ **.**

Using the time delay property,

$$x_2(t)=-F_0\left(\frac{1}{k}+\frac{e^{a_2(t-t_0)}}{3ma_2^2+2ca_2+k}+\frac{e^{a_3(t-t_0)}}{3ma_3^2+2ca_3+k}\right) \quad t>t_0,\ x_2(t)=0 \quad \text{for } t<t_0$$

Laplace Transform Method: Example 3



$m = 10$ kg

$c = 50$ N-sec/m

$k = 1000$ N/m